Ehrhart's sharp volume bound

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August 5, 2026*

Theorem 1. *Let $K \subseteq \mathbb{R}^n$ be a full-dimensional compact convex body whose barycenter is the origin, and suppose that the origin is the only lattice point in the interior of K:*

$$\operatorname{int}(K) \cap \mathbb{Z}^n = \{0\}.$$

Then

$$\operatorname{vol}(K) \leq \frac{(n+1)^n}{n!}.$$

The constant is sharp: equality holds for the centered simplex $(n+1)\Delta_n - (1, \ldots, 1)$, where Δ_n is the standard n-dimensional simplex.

The proof combines a jet-counting lower bound with reduction modulo a prime: both are applied to a product of basis elements simultaneously adapted to the order-of-vanishing filtration at $(1, \ldots, 1)$ and to a generic monomial ordering, whose Newton polytope has the origin as an exposed vertex.

Notation. Throughout, vol is Lebesgue measure on $\mathbb{R}^n$, normalized so that $\mathbb{Z}^n$ has covolume one, and $g(K) = \operatorname{vol}(K)^{-1} \int_K x \, dx$ is the barycenter of a full-dimensional compact convex body K. We write $\Delta_n = \operatorname{conv}\{0, e_1, \ldots, e_n\}$, $(x)_+ = \max(x, 0)$, and $\mathbf{1} = (1, \ldots, 1)$. For $u \in \mathbb{Z}^n$, $z^u = z_1^{u_1} \cdots z_n^{u_n}$ denotes a Laurent monomial; the support $\operatorname{supp}(f)$ of a Laurent polynomial f is its set of exponents with nonzero coefficient, and $\operatorname{Newt}(f)$ is the convex hull of $\operatorname{supp}(f)$.

1 Reduction to centered rational polytopes

Lemma 2. *If a full-dimensional compact convex body K has barycenter zero, then $0 \in \operatorname{int}(K)$.*

Proof. If 0 were not interior, separating a point of $\mathbb{R}^n \setminus \operatorname{int}(K)$ from the convex set $\operatorname{int}(K)$ would give a nonzero linear functional ℓ with $\ell \geq 0$ on K. Full dimensionality makes $\ell > 0$ on a relatively open subset of K of positive volume, whereas $\ell \geq 0$ everywhere on K; hence $\ell(\int_K x \, dx) > 0$, contradicting $g(K) = 0$. □

The heart of the matter is the following special case.

Proposition 3. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional rational polytope with*

$$g(P) = 0, \qquad \operatorname{int}(P) \cap \mathbb{Z}^n = \{0\}.$$

Then $\operatorname{vol}(P) \leq (n+1)^n/n!$.

*Updated August 10, 2026: proof of Lemma 5 fixed.

Sections 2–6 prove Proposition 3; here we record that it implies the theorem.

Lemma 4. *Proposition 3 implies the volume bound of Theorem 1 for every body K as in the theorem.*

Proof. By Lemma 2, $0 \in \operatorname{int}(K)$, so $\varepsilon B \subseteq K$ for some $\varepsilon > 0$, where B is the closed unit ball. Fix $0 < \delta < 1$. By convexity, $(1-\delta)K + \delta\varepsilon B \subseteq K$, so $(1-\delta)K \subseteq \operatorname{int}(K)$; the body $(1-\delta)K$ is again centered. Only finitely many nonzero lattice points lie within distance 1 of the compact set $(1-\delta)K$; none of them belongs to it, since $(1-\delta)K \subseteq \operatorname{int}(K)$ meets $\mathbb{Z}^n$ only in 0. Hence there is a margin $\rho > 0$ such that every nonzero lattice point is at distance at least ρ from $(1-\delta)K$.

Approximate $(1-\delta)K$ in Hausdorff distance by a full-dimensional rational polytope Q'. Barycenters of full-dimensional convex bodies vary continuously under Hausdorff convergence, so $g(Q') \to 0$ as the approximation improves; moreover the barycenter of a rational polytope is rational (triangulate it into rational simplices and use their rational volumes and centroids). Thus

$$P = Q' - g(Q')$$

is a rational polytope with $g(P) = 0$, arbitrarily close to $(1-\delta)K$ in Hausdorff distance. For a sufficiently close approximation, P is full-dimensional with $0 \in \operatorname{int}(P)$, and every nonzero lattice point stays at distance at least $\rho/2$ from P; that is, $\operatorname{int}(P) \cap \mathbb{Z}^n = \{0\}$.

Proposition 3 gives $\operatorname{vol}(P) \leq (n+1)^n/n!$. Volume is continuous under Hausdorff convergence of full-dimensional compact convex bodies, so passing to the approximation limit yields $(1-\delta)^n \operatorname{vol}(K) = \operatorname{vol}\big((1-\delta)K\big) \leq (n+1)^n/n!$, and letting $\delta \downarrow 0$ finishes the proof. ☐

2 Jets of Laurent polynomials at 1

Fix a field F. Since $1 + y_i$ is a unit in the formal power series ring $F[[y_1, \ldots, y_n]]$, the universal property of the Laurent polynomial ring provides a unique F-algebra homomorphism

$$E\colon F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \longrightarrow F[[y_1, \ldots, y_n]], \qquad E(z_i) = 1 + y_i,$$

the Taylor expansion at $\mathbf{1}$ in the coordinates $y_i = z_i - 1$. On a monomial z^u, $u \in \mathbb{Z}^n$, the coefficient of y^α in $E(z^u)$ is

$$\prod_{i=1}^{n} \binom{u_i}{\alpha_i}, \qquad \binom{u}{a} = \frac{u(u-1)\cdots(u-a+1)}{a!},$$

an *integer* even when some u_i are negative.

The map E is injective: given $f \neq 0$, multiply by a monomial z^v so that $z^v f \in F[z_1, \ldots, z_n]$; the substitution $z_i \mapsto 1 + y_i$ is an automorphism of the polynomial ring, so $E(z^v f) \neq 0$, and $E(z^v) = \prod_i (1+y_i)^{v_i}$ is a unit, whence $E(f) \neq 0$. In other words, a nonzero Laurent polynomial cannot have zero Taylor series at a point.

For $f \neq 0$ let $\operatorname{ord}_{\mathbf{1}}(f)$ be the least total degree of a nonzero coefficient of $E(f)$; by injectivity, $\operatorname{ord}_{\mathbf{1}}(f) < \infty$. Orders add under multiplication,

$$\operatorname{ord}_{\mathbf{1}}(fg) = \operatorname{ord}_{\mathbf{1}}(f) + \operatorname{ord}_{\mathbf{1}}(g),$$

because the initial forms (lowest-degree homogeneous parts) multiply in the polynomial ring $F[y_1, \dots, y_n]$, which is a domain.

Now let P be a polytope as in Proposition 3. For $k \geq 1$ set

$$A_k = kP \cap \mathbb{Z}^n, \qquad N_k = \#A_k, \qquad V_k^F = \operatorname{span}_F\{z^u : u \in A_k\}.$$

Distinct monomials are linearly independent, so $\dim V_k^F = N_k$. Define the vanishing filtration and its dimension data

$$W_{k,m}^F = \{f \in V_k^F : f = 0 \text{ or } \operatorname{ord}_{\mathbf{1}}(f) \geq m\}, \qquad d_{k,m}^F = \dim W_{k,m}^F, \qquad S_k^F = \sum_{m \geq 1} d_{k,m}^F.$$

The sum is finite: $W_{k,m}^F = 0$ for large m, again by injectivity of E (a nonzero element of the finite-dimensional space V_k^F has finite order, and there are only finitely many linearly independent constraints below any fixed order).

Concretely, the number of monomials y^α with total degree $|\alpha| \leq m - 1$ is

$$J_m = \binom{n + m - 1}{n},$$

and $W_{k,m}^F$ is the kernel of the linear *jet map* $V_k^F \to F^{J_m}$ sending f to the coefficients of $E(f)$ in total degree $< m$. In the monomial basis of V_k^F this map is represented by the *jet matrix* with entries $\prod_i \binom{u_i}{\alpha_i}$ — an integer matrix independent of F. Consequently

$$d_{k,m}^F = N_k - \operatorname{rank}_F(\text{jet matrix}),$$

so $d_{k,m}^F$ and S_k^F depend only on the characteristic of F; we write $d_{k,m}^{(0)}$, S_k in characteristic zero and $d_{k,m}^{(p)}$, $S_{k,p}$ over $\mathbb{F}_p$.

Finally, we call a basis $f_1, \dots, f_{N_k}$ of V_k^F *adapted* to the vanishing filtration if $W_{k,m}^F = \operatorname{span}\{f_i : \operatorname{ord}_{\mathbf{1}}(f_i) \geq m\}$ for every $m \geq 1$; then $\#\{i : \operatorname{ord}_{\mathbf{1}}(f_i) \geq m\} = d_{k,m}^F$, so

$$\sum_{i=1}^{N_k} \operatorname{ord}_{\mathbf{1}}(f_i) = \sum_{m \geq 1} d_{k,m}^F = S_k^F.$$

3 The jet lower bound

Work in characteristic zero. Since $\dim V_k^F = N_k$ and the jet map has at most J_m independent components,

$$d_{k,m}^{(0)} \geq (N_k - J_m)_+,$$

and hence

$$S_k \geq \sum_{m \geq 1} \left(N_k - \binom{n + m - 1}{n} \right)_+.$$

The lattice Riemann-sum estimate for a rational polytope gives

$$N_k = \operatorname{vol}(P) k^n + O(k^{n-1})$$

(triangulate the boundary of P into finitely many Lipschitz graphs and compare with unit lattice cubes at scale $1/k$; the boundary layer meets $O(k^{n-1})$ cubes), while

$$\binom{n+m-1}{n} = \frac{m^n}{n!} + O(m^{n-1}), \quad \text{uniformly for } m = O(k).$$

Writing $M_k = (n!N_k)^{1/n}$, so that $N_k - m^n/n!$ changes sign at $m = M_k = O(k)$, comparison of the sum with its Riemann integral yields

$$\sum_{m\geq 1}\Big(N_k - \binom{n+m-1}{n}\Big)_+ = N_kM_k - \frac{M_k^{n+1}}{(n+1)!} + O(k^n) = \frac{n}{n+1}N_kM_k + O(k^n),$$

whence

$$S_k \geq \frac{n}{n+1}N_k(n!N_k)^{1/n} - O(k^n).$$

Dividing by $kN_k \asymp k^{n+1}$ and letting $k \to \infty$,

$$\liminf_{k\to\infty} \frac{S_k}{kN_k} \geq \frac{n}{n+1}\big(n!\,\mathrm{vol}(P)\big)^{1/n}. \tag{1}$$

4 Adapted products and an exposed Newton vertex

Fix a linear functional $\ell\colon \mathbb{R}^n \to \mathbb{R}$ that is injective on the finite set A_k (a generic functional has this property). For $f \in V_k^F$, $f \neq 0$, call the ℓ-minimal element of $\mathrm{supp}(f)$ the *pivot* of f; it is unique because ℓ is injective on $A_k \supseteq \mathrm{supp}(f)$.

Lemma 5. *Over any field F there is a basis $f_1, \ldots, f_{N_k}$ of V_k^F, adapted to the vanishing filtration, whose pivots are pairwise distinct — hence run through all of A_k.*

Proof. Adapted bases exist. Set $W_{k,0}^F = V_k^F$ and recall from Section 2 that $W_{k,m}^F = 0$ for large m. Choosing for every $m \geq 0$ a complement C_m of $W_{k,m+1}^F$ in $W_{k,m}^F$ gives $V_k^F = \bigoplus_{m\geq 0} C_m$ with $W_{k,m}^F = \bigoplus_{m'\geq m} C_{m'}$, and every nonzero element of C_m has order exactly m (order at least $m+1$ would put it in $C_m \cap W_{k,m+1}^F = 0$); the union of bases of the C_m is therefore an adapted basis.

We record a counting criterion: a basis $g_1, \ldots, g_{N_k}$ of V_k^F with $\#\{i : \mathrm{ord}_{\mathbf{1}}(g_i) \geq m\} = d_{k,m}^F$ for every $m \geq 1$ is adapted, because the g_i of order at least m are linearly independent vectors of the $d_{k,m}^F$-dimensional space $W_{k,m}^F$ and hence span it. Conversely, every adapted basis has these counts (Section 2).

Now make the pivots distinct by Gaussian elimination. Suppose two elements $f_i \neq f_j$ of an adapted basis share the pivot u, indexed so that $\mathrm{ord}_{\mathbf{1}}(f_i) \leq \mathrm{ord}_{\mathbf{1}}(f_j)$, and let $c \in F^\times$ be the ratio of their coefficients at z^u. Replace f_i by $f' = f_i - cf_j$. This elementary operation returns a basis, and $f' \neq 0$. Its order satisfies $\mathrm{ord}_{\mathbf{1}}(f') \geq \min\big(\mathrm{ord}_{\mathbf{1}}(f_i), \mathrm{ord}_{\mathbf{1}}(f_j)\big) = \mathrm{ord}_{\mathbf{1}}(f_i)$, and equality must hold: otherwise the new basis would contain $d_{k,m}^F + 1$ linearly independent elements of order at least $m = \mathrm{ord}_{\mathbf{1}}(f_i) + 1$ inside $W_{k,m}^F$. The counts $\#\{i : \mathrm{ord}_{\mathbf{1}}(f_i) \geq m\}$ are therefore unchanged, so the new basis is still adapted by the criterion above. Moreover $\mathrm{supp}(f') \subseteq \big(\mathrm{supp}(f_i) \cup \mathrm{supp}(f_j)\big) \setminus \{u\}$, and u was the ℓ-minimal point of both supports, so the pivot of f' has strictly larger ℓ-value than u.

Each replacement strictly increases $\sum_i \ell(\text{pivot of } f_i)$, which takes values in the finite set of N_k-term sums of elements of $\ell(A_k)$; hence after finitely many replacements no two basis elements share a pivot. The N_k pairwise distinct pivots lie in the N_k-element set A_k and therefore exhaust it. $\square$

Let $f_1, \dots, f_{N_k}$ be as in Lemma 5 and put

$$F_k = \prod_{i=1}^{N_k} f_i, \qquad U_k = \sum_{u \in A_k} u, \qquad G_k := z^{-U_k} F_k.$$

Since orders add and the basis is adapted,

$$\operatorname{ord}_{\mathbf{1}} G_k = \operatorname{ord}_{\mathbf{1}} F_k = \sum_{i=1}^{N_k} \operatorname{ord}_{\mathbf{1}}(f_i) = S_k^F. \tag{2}$$

Moreover *zero is an exposed exponent of G_k with nonzero coefficient.* Indeed, the coefficient of z^{U_k} in F_k is the product of the pivot coefficients of the f_i: any decomposition $U_k = \sum_i u_i'$ with $u_i' \in \operatorname{supp}(f_i)$ satisfies $\ell(u_i') \geq \ell(\text{pivot of } f_i)$ with equality only for the pivot itself; summing, and using that the pivots are exactly the elements of A_k (so their sum is U_k), every u_i' must be the pivot of f_i. Thus z^{U_k} is the unique ℓ-minimal term of F_k, its coefficient does not cancel, and U_k is exposed (by the functional ℓ) in $\operatorname{Newt}(F_k)$; after the translation by $-U_k$, the origin is an exposed vertex of $Q_k := \operatorname{Newt}(G_k)$.

Finally, every exponent of F_k is a sum of N_k points of the convex set kP and therefore lies in $N_k \cdot kP = kN_kP$, so

$$Q_k = \operatorname{Newt}(G_k) \subseteq kN_kP - U_k. \tag{3}$$

Note that Lemma 5 and the statements (2)–(3) hold verbatim over every field F.

5 The radius bound

For a compact set $Q \subseteq \mathbb{R}^n$ define

$$L(Q, 0) = \max\Big(\{m \in \mathbb{Z}_{\geq 0} : mz \in Q \text{ for some } z \in \mathbb{Z}^n \setminus \{0\}\} \cup \{0\}\Big);$$

the maximum is finite because Q is bounded and $|z| \geq 1$. Clearly $L(\cdot, 0)$ is monotone under inclusion.

Since the barycenter of P is zero, the first-moment lattice Riemann sum (the boundary-cube comparison of Section 3, applied coordinatewise) gives

$$\frac{1}{k \operatorname{vol}(P) k^n} \sum_{u \in A_k} u \longrightarrow \frac{1}{\operatorname{vol}(P)} \int_P x \, dx = 0,$$

and $N_k \sim \operatorname{vol}(P) k^n$, so

$$b_k := \frac{U_k}{kN_k} \longrightarrow 0. \tag{4}$$

Lemma 6. *Set $\Lambda_k = L\big(kN_kP - U_k, 0\big)$. For every $\varepsilon > 0$ there is $k_0 = k_0(\varepsilon, P)$ such that $\Lambda_k \leq (1+\varepsilon)kN_k$ for all $k \geq k_0$. Consequently, by monotonicity,*

$$L(Q, 0) \leq (1+\varepsilon)kN_k \qquad \textit{for } k \geq k_0 \textit{ and } \text{every } \textit{polytope } Q \subseteq kN_kP - U_k. \tag{5}$$

Proof. Fix $R \geq 1$ with $P \subseteq RB$ and, using $0 \in \operatorname{int}(P)$, fix $\rho > 0$ with $\rho B \subseteq P$. Choose k_0 so large that $|b_k| \leq \min(1, \varepsilon\rho/2)$ for $k \geq k_0$, which is possible by (4).

Suppose, for some $k \geq k_0$, that $mz \in kN_kP - U_k$ with $z \in \mathbb{Z}^n \setminus \{0\}$ and $t := m/(kN_k) \geq 1+\varepsilon$. Dividing by kN_k,

$$b_k + tz \in P.$$

Put

$$q = -\frac{b_k}{t-1}, \qquad \text{so that} \qquad z = \frac{1}{t}\,(b_k + tz) + \Big(1 - \frac{1}{t}\Big)q.$$

Since $t - 1 \geq \varepsilon$, we get $|q| \leq |b_k|/\varepsilon \leq \rho/2$, so $q \in \operatorname{int}(P)$. The displayed identity exhibits z as a convex combination of the point $b_k + tz \in P$ and the interior point q, with strictly positive weight $1 - 1/t \geq \varepsilon/(1+\varepsilon)$ on q; by the standard convexity fact $\lambda x + (1-\lambda)q \in \operatorname{int}(P)$ for $x \in P$, $q \in \operatorname{int}(P)$, $0 \leq \lambda < 1$, we conclude $z \in \operatorname{int}(P)$. Thus $z \in \operatorname{int}(P) \cap \mathbb{Z}^n \setminus \{0\}$, contradicting the lattice hypothesis. Hence $t < 1 + \varepsilon$ for every such m and z, which is the claim. □

The threshold $k_0(\varepsilon, P)$ depends only on ε and P — through R, ρ, and the convergence (4) — and in particular not on the field F, the basis of Lemma 5, or the polytope Q in (5). This uniformity is what permits choosing the prime p in the next section *before* constructing the $\mathbb{F}_p$-basis.

6 Reduction modulo p

The jet matrices of Section 2 are integer matrices, and reduction modulo a prime p cannot increase the rank of an integer matrix. Hence

$$d^{(p)}_{k,m} = N_k - \operatorname{rank}_{\mathbb{F}_p}(\text{jet matrix}) \geq N_k - \operatorname{rank}_{\mathbb{Q}}(\text{jet matrix}) = d^{(0)}_{k,m} \qquad \text{for every } m,$$

and both sums are finite (Section 2), so

$$S_k \leq S_{k,p} = \sum_{m \geq 1} d^{(p)}_{k,m}. \tag{6}$$

Apply Lemma 5 over $F = \mathbb{F}_p$ — the basis is now adapted to the $\mathbb{F}_p$ vanishing filtration, whose dimension data is $d^{(p)}_{k,m}$ — and form F_k, U_k, G_k, Q_k as in Section 4. By (2)–(3), which hold over every field,

$$\operatorname{ord}_{\mathbf{1}} G_k = S_{k,p},$$

zero is an exposed exponent of G_k with nonzero coefficient, and $Q_k \subseteq kN_kP - U_k$.

Suppose now that $p > L(Q_k, 0)$, and consider the ring

$$R_p = \mathbb{F}_p[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]/(z_1^p - 1, \ldots, z_n^p - 1).$$

In characteristic p we have $z_i^p - 1 = (z_i - 1)^p$, so in the coordinates $y_i = z_i - 1$,

$$R_p \simeq \mathbb{F}_p[y_1, \ldots, y_n]/(y_1^p, \ldots, y_n^p).$$

Every surviving monomial y^α has each $\alpha_i \leq p - 1$, hence total degree at most $n(p-1)$; the maximal ideal $\mathfrak{m} = (y_1, \ldots, y_n)$ therefore satisfies

$$\mathfrak{m}^{n(p-1)+1} = 0. \tag{7}$$

The canonical quotient map $\mathbb{F}_p[z^{\pm 1}] \to R_p$ coincides with the composite

$$\mathbb{F}_p[z^{\pm 1}] \xrightarrow{E} \mathbb{F}_p[[y_1, \dots, y_n]] \longrightarrow \mathbb{F}_p[[y_1, \dots, y_n]]/(y_1^p, \dots, y_n^p) \simeq R_p :$$

both are $\mathbb{F}_p$-algebra homomorphisms sending $z_i \mapsto 1 + y_i$, and a homomorphism out of the Laurent polynomial ring is determined by the (invertible) images of the z_i. In particular, each negative Laurent power expands in R_p as its finite power series in the y_i. Consequently, if $\operatorname{ord}_{\mathbf{1}} G_k > n(p-1)$, then every coefficient of $E(G_k)$ in total degree $\le n(p-1)$ vanishes, so the image of G_k in R_p lies in $\mathfrak{m}^{n(p-1)+1} = 0$ by (7).

On the other hand, the residue class of z^u in R_p depends only on u mod p, and the classes of the monomials z^u, $u \in \{0, \dots, p-1\}^n$, form a basis of R_p, indexed by $(\mathbb{Z}/p\mathbb{Z})^n$. No exponent of G_k other than zero is congruent to zero modulo p: such an exponent would equal pz for some $z \in \mathbb{Z}^n \setminus \{0\}$, and $pz \in Q_k$ would give $L(Q_k, 0) \ge p$, contradicting $p > L(Q_k, 0)$. Hence the coefficient of the image of G_k on the basis element of residue class zero is exactly the nonzero exposed coefficient of G_k, so the image of G_k in R_p is *nonzero*. Combining the two paragraphs,

$$S_{k,p} = \operatorname{ord}_{\mathbf{1}} G_k \le n(p-1) \qquad \text{whenever } p > L(Q_k, 0). \tag{8}$$

It remains to choose p. We use the following standard consequence of the prime number theorem: for every fixed $\eta > 0$, every sufficiently large real x admits a prime in $(x, (1+\eta)x)$, since $\pi\big((1+\eta)x\big) - \pi(x) \sim \eta x/\log x \to \infty$. Applying this with $x = (1+\eta)kN_k$, choose for every large k a prime

$$(1+\eta)kN_k < p < (1+\eta)^2 kN_k.$$

By Lemma 6 with $\varepsilon = \eta$ — whose bound (5) holds uniformly for every polytope contained in $kN_kP - U_k$, in particular for Q_k, whatever basis produced it — we have $L(Q_k, 0) \le (1+\eta)kN_k < p$ for all $k \ge k_0(\eta, P)$. Then (6) and (8) give

$$\frac{S_k}{kN_k} \le \frac{S_{k,p}}{kN_k} \le \frac{n(p-1)}{kN_k} < n(1+\eta)^2,$$

so $\limsup_{k\to\infty} S_k/(kN_k) \le n(1+\eta)^2$. Letting $\eta \downarrow 0$,

$$\limsup_{k\to\infty} \frac{S_k}{kN_k} \le n. \tag{9}$$

Combining (1) and (9),

$$\frac{n}{n+1}\big(n! \operatorname{vol}(P)\big)^{1/n} \le n,$$

hence

$$\operatorname{vol}(P) \le \frac{(n+1)^n}{n!}\,. \tag{10}$$

This proves Proposition 3, and with Lemma 4 the volume bound of Theorem 1.

7 Sharpness

Let $S = (n+1)\Delta_n - \mathbf{1}$. Then $\operatorname{vol}(S) = (n+1)^n \operatorname{vol}(\Delta_n) = (n+1)^n/n!$. The barycenter of a simplex is the average of its vertices, so

$$g\big((n+1)\Delta_n\big) = \frac{0 + (n+1)e_1 + \cdots + (n+1)e_n}{n+1} = \mathbf{1},$$

and the translate S has barycenter zero. Finally, a point $x \in \mathrm{int}(S)$ satisfies $x_i > -1$ for all i and $\sum_i (x_i + 1) < n + 1$; for $x \in \mathbb{Z}^n$ the integers $y_i = x_i + 1 \geq 1$ then satisfy $\sum_i y_i < n + 1$, forcing $y_i = 1$ for all i, that is, $x = 0$. Thus S satisfies the hypotheses of Theorem 1 with $\mathrm{vol}(S) = (n + 1)^n / n!$, and the constant is sharp. $\square$

8 AI-assisted work

This work — research, development, verification, writing, and review — was produced with extensive use of AI systems, including large language models.