Two Proofs of the Sharp Inequality in Ehrhart's Volume Conjecture: Verification and Comparison

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Abstract

We are releasing a new proof of Ehrhart's volume conjecture: the sharp bound $\mathrm{vol}(K) \leq (n+1)^n/n!$ for a convex body $K \subseteq \mathbb{R}^n$ whose barycenter is its unique interior lattice point. The proof itself appears in an accompanying manuscript; the present document is its companion, explaining the new argument, recording its verification, and comparing it with the proof recently published by OpenAI. While OpenAI's proof is analytic—built on a real Monge–Ampère transport potential, weighted Bergman spaces on the complex torus, and Berndtsson's positivity theorem—our approach is algebraic and largely elementary: a jet-dimension count at $(1, \ldots, 1)$, a product of an adapted basis with an exposed Newton vertex, and reduction modulo a well-chosen prime, where nilpotency in a truncated group algebra caps the possible vanishing. The two proofs share their easy half: both pinch a normalized vanishing order between $\frac{n}{n+1}(n!\,\mathrm{vol})^{1/n}$ and n by a filtration count going back to Fujita. Their hard halves are disjoint: the hypotheses enter through different objects, the constant n arises from different sources, and no step of one argument translates to a step of the other. We have verified the new proof in full—every step audited, its mechanism corroborated by exact computation—and found it correct and self-contained, with the prime number theorem as its deepest external input. Neither proof classifies the equality cases; that classification has since been established by Liu (arXiv:2608.01040), within the analytic framework.

1 Introduction

Throughout, a *convex body* is a full-dimensional compact convex subset of $\mathbb{R}^n$, vol is Lebesgue measure normalized so that $\mathbb{Z}^n$ has covolume one, $g(K) = \mathrm{vol}(K)^{-1} \int_K x\, dx$ is the barycenter, and $\Delta_n = \mathrm{conv}\{0, e_1, \ldots, e_n\}$ is the standard simplex. Both proofs under discussion establish exactly the following statement.

Theorem 1 (Ehrhart's volume conjecture)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex body with $g(K) = 0$ and*

$$\mathrm{int}(K) \cap \mathbb{Z}^n = \{0\}.$$

Then

$$\mathrm{vol}(K) \leq \frac{(n+1)^n}{n!}\,.$$

The constant is sharp: equality holds for the centered simplex $(n+1)\Delta_n - (1, \ldots, 1)$.

History. Ehrhart posed the conjecture in 1964 [Ehr64] and proved it for planar bodies and for simplices in every dimension [Ehr55, Ehr79]. For arbitrary centered bodies, symmetrization and Minkowski's theorem give $\mathrm{vol}(K) \le 4^n$ [MP00, HHH16]; successive improvements reached $4^n e^{-c\sqrt{n}}$ via thin-shell estimates [HSTV22], then $4^n \exp(-cn/(\log n)^8)$ via bounds on the isotropic constant [CHMT24], and, combining the latter argument with the resolution of Bourgain's slicing problem [KL25], $4^n e^{-cn}$. The sharp constant was known for several classes of bodies: Berman and Berndtsson proved it for centered rational polytopes whose facet presentation $\{y : \langle \ell_F, y\rangle \ge -a_F\}$ has primitive integer normals and $0 < a_F \le 1$, and for bodies in the positive orthant with barycenter $(1, \ldots, 1)$ [BB17]; Nill and Paffenholz proved it for bodies contained in the polar of a lattice polytope and classified the equality cases under that hypothesis [NP14]. The general case remained open for six decades.

The two proofs. In August 2026 OpenAI released a complete proof of Theorem 1 as Chapter 8 of *Ten Advances in Mathematics and Theoretical Computer Science* [OAI26], accompanied by discovery notes [OAIn26] and a Lean certificate [OAIL26]. That proof is analytic: it encodes K in a real Monge–Ampère transport potential [BB13], realizes the lattice points of dilates of K as an orthogonal basis of weighted Bergman spaces on the complex torus, and obtains the decisive upper bound from Berndtsson's positivity theorem [Ber06] together with a Schwarz-lemma estimate on a shrinking complex ball.

Our manuscript [Init26], released alongside this note, proves the same theorem by an essentially algebraic route: a jet-dimension count at the point $\mathbf{1} = (1, \ldots, 1)$, a product of an adapted basis whose Newton polytope has the origin as an exposed vertex, and reduction modulo a prime p chosen—by the prime number theorem—just above the diameter scale of that Newton polytope, where the truncated group algebra $\mathbb{F}_p[y_1, \ldots, y_n]/(y_1^p, \ldots, y_n^p)$ caps the possible vanishing.

What this note contains. Section 2 isolates the skeleton common to the two proofs. Sections 3 and 4 explain the algebraic and the analytic argument in turn, at a level of detail meant to make the comparison in Section 5 checkable against the sources. Section 6 records our verification of [Init26], including exact-arithmetic computations that corroborate its mechanism at finite level. Section 7 discusses what each proof offers that the other does not. Our conclusions, in brief:

- The manuscript [Init26] is mathematically correct and self-contained. Its external inputs are classical and are named where used; the deepest is the prime number theorem, and only the sharpness of the constant requires it (Remark 3).
- The two proofs share their lower half exactly: the same vanishing-order filtration, at the same point of the same torus, with the same codimension count and the same sharp integral—a mechanism that goes back to Fujita's work in Fano geometry [Fuj18], as Chapter 8 notes explicitly.
- The upper halves are disjoint. The two hypotheses of Theorem 1 enter through different objects, the rigidity mechanisms share no machinery, and the constant n arises from different sources. We found no step-by-step translation between them (Section 5.3).
- Neither proof determines whether the extremizers are exactly the unimodular images of $(n+1)\Delta_n - \mathbf{1}$, as conjectured by Nill and Paffenholz [NP14, Conj. 1.1]. As this note was being completed, Liu [Liu26] established that classification, building on the analytic framework of [OAI26]; we discuss its bearing on the comparison in Section 7.

This note is expository and evaluative; all theorems discussed are proved in [Init26] and [OAI26].

2 The common skeleton

Both proofs are slope-pinch arguments, and they pinch the same quantity. Fix the point $\mathbf{1} = (1, \dots, 1)$ of the algebraic torus. To a growing family of finite-dimensional spaces of functions with lattice exponents—Laurent polynomials in [Init26], weighted-square-integrable holomorphic functions in [OAI26]—one attaches the filtration by order of vanishing at $\mathbf{1}$ and forms the *normalized aggregate vanishing order*: the average, over a basis adapted to the filtration, of (order of basis element)/k at level k. In the notation of the two sources this quantity is

$$\sigma_k = \frac{S_k}{kN_k} = \frac{1}{N_k}\sum_{i=1}^{N_k}\frac{\operatorname{ord}_{\mathbf{1}}(f_i)}{k} \qquad \text{and} \qquad \widetilde{\sigma}_k = \int_X g_k\, d\mu_k = \frac{1}{d_k}\sum_{a=1}^{d_k}\frac{q_a}{k}\,,$$

respectively, where $f_1, \dots, f_{N_k}$ (resp. $s_1, \dots, s_{d_k}$) is an adapted basis and q_a are truncated vanishing orders; the notation is recalled in Sections 3 and 4. Each proof establishes, for its own version of this quantity, the two asymptotic bounds

$$\frac{n}{n+1}(n!\operatorname{vol})^{1/n} \;\le\; \Big(\text{normalized aggregate vanishing order at } \mathbf{1}\Big) \;\le\; n, \tag{2.1}$$

and comparison of the two sides gives $(n!\operatorname{vol})^{1/n} \le n+1$, which is Theorem 1.

The lower halves are not merely analogous; they are the same argument. Vanishing to order $\ge m$ at a smooth point of an n-dimensional space imposes at most

$$J_m = \binom{n+m-1}{n}$$

linear conditions, so a space of dimension N contains a subspace of dimension $\ge (N - J_m)_+$ vanishing to order $\ge m$. Summing over m, using the lattice-point asymptotics $N \sim \operatorname{vol}\cdot k^n$, and comparing with an integral yields

$$\frac{1}{\operatorname{vol}}\int_0^c \Big(\operatorname{vol} - \frac{s^n}{n!}\Big) ds = \frac{n}{n+1}\, c, \qquad c = (n!\operatorname{vol})^{1/n},$$

the left-hand bound of (2.1). This count and this integral appear as equation (1) of [Init26] and as equation (18) of [OAI26]; Chapter 8 traces them to Fujita's point-filtration proof of the sharp volume bound for K-semistable Fano manifolds [Fuj18, Thms. 2.3 and 5.1], and the same lineage applies verbatim to [Init26]. Both proofs also verify sharpness by the same two-line computation on the simplex $(n+1)\Delta_n - \mathbf{1}$.

Everything else—the entire upper half of (2.1), which is where the barycenter and lattice hypotheses must do their work—is different, and Sections 3–5 are devoted to how.

Remark 1 (Notation). The two sources use overlapping symbols with different meanings: in [Init26], $N_k = \#(kP \cap \mathbb{Z}^n)$ and $L(Q, 0)$ is a lattice radial function, while in [OAI26], N_k denotes the truncation level $\lfloor c_K k \rfloor$ and $L(t)$ a logarithmic partition function. We keep each source's notation within the section devoted to it, write $d_k = \dim \mathcal{H}_k$ throughout, and avoid the symbol N_k in Section 4.

3 The algebraic proof

We summarize [Init26], following its structure; the labels (1)–(10) and Lemmas 2–6 below refer to that manuscript.

3.1 Reduction to centered rational polytopes

A convex body with barycenter zero has 0 in its interior (Lemma 2). The theorem is reduced (Lemma 4) to the case of a centered rational polytope P with $\operatorname{int}(P) \cap \mathbb{Z}^n = \{0\}$: shrink K to $(1-\delta)K \subseteq \operatorname{int}(K)$, note that the nonzero lattice points keep a positive distance ρ from the shrunken body, approximate within $\rho/2$ in Hausdorff distance by a rational polytope, and recenter by the (rational) barycenter. Volume continuity and $\delta \downarrow 0$ recover the bound for K. All facts used—barycenter continuity, rationality of the barycenter of a rational polytope, volume continuity—are elementary and proved or referenced in the manuscript.

3.2 Jets and the lower bound

For a field F, Taylor expansion at $\mathbf{1}$ in the coordinates $y_i = z_i - 1$ is the algebra map $E\colon F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to F[[y_1, \ldots, y_n]]$, $z_i \mapsto 1 + y_i$. On a monomial z^u the coefficient of y^α is $\prod_i \binom{u_i}{\alpha_i}$, an *integer* even for negative u_i. The map E is injective, orders add under multiplication, and for

$$A_k = kP \cap \mathbb{Z}^n, \qquad N_k = \# A_k, \qquad V_k^F = \operatorname{span}_F\{z^u : u \in A_k\},$$

the filtration dimensions $d_{k,m}^F = \dim\{f \in V_k^F : \operatorname{ord}_{\mathbf{1}} f \geq m\}$ are corank data of *integer* jet matrices. Hence they depend only on $\operatorname{char} F$; write $d_{k,m}^{(0)}, S_k$ in characteristic zero and $d_{k,m}^{(p)}, S_{k,p}$ over $\mathbb{F}_p$, where $S_k^F = \sum_{m \geq 1} d_{k,m}^F$ is the aggregate order. Rank–nullity gives $d_{k,m}^{(0)} \geq (N_k - J_m)_+$, and lattice Riemann sums turn this into the lower half of the pinch,

$$\liminf_{k \to \infty} \frac{S_k}{k N_k} \;\geq\; \frac{n}{n+1} \big(n! \operatorname{vol}(P)\big)^{1/n}. \tag{1}$$

3.3 The adapted product and its exposed Newton vertex

Fix a linear functional ℓ injective on A_k and call the ℓ-minimal exponent of $f \neq 0$ its *pivot*. Lemma 5 (a two-flag, or Bruhat, argument: refine the vanishing filtration to a complete flag and intersect with the flag of ℓ-initial monomial segments) produces a basis $f_1, \ldots, f_{N_k}$ of V_k^F that is simultaneously adapted to the vanishing filtration and has pairwise distinct pivots—which therefore exhaust A_k. Put

$$F_k = \prod_{i=1}^{N_k} f_i, \qquad U_k = \sum_{u \in A_k} u, \qquad G_k = z^{-U_k} F_k.$$

Additivity of orders gives $\operatorname{ord}_{\mathbf{1}} G_k = \sum_i \operatorname{ord}_{\mathbf{1}}(f_i) = S_k^F$ (equation (2)). The coefficient of z^{U_k} in F_k is the product of the pivot coefficients and cannot cancel: in any decomposition $U_k = \sum_i u_i'$ with $u_i' \in \operatorname{supp}(f_i)$, each $\ell(u_i')$ is at least $\ell(\mathrm{pivot}_i)$ with equality only at the pivot, and the pivots sum to exactly U_k. Hence after the shift by $-U_k$ *the origin is an exposed vertex, with nonzero coefficient, of* $Q_k = \mathrm{Newt}(G_k)$, and since every exponent of F_k is a sum of N_k points of kP,

$$Q_k \subseteq kN_kP - U_k. \tag{3}$$

All of this holds verbatim over every field.

3.4 The radius bound

Both hypotheses of the theorem now enter, through elementary convexity. For compact $Q \subseteq \mathbb{R}^n$ let $L(Q, 0)$ be the largest $m \geq 0$ with $mz \in Q$ for some $z \in \mathbb{Z}^n \setminus \{0\}$; it is monotone under inclusion. The barycenter hypothesis enters as a first-moment Riemann sum: $b_k = U_k/(kN_k) \to 0$ (equation (4)). The lattice hypothesis enters in Lemma 6: for every $\varepsilon > 0$ there is $k_0(\varepsilon, P)$ with

$$L(Q, 0) \leq (1 + \varepsilon)kN_k \qquad \text{for all } k \geq k_0 \text{ and } \textit{every } Q \subseteq kN_kP - U_k. \tag{5}$$

The proof is a three-line contradiction: if $mz \in kN_kP - U_k$ with $t = m/(kN_k) \geq 1 + \varepsilon$, then $b_k + tz \in P$, and writing z as the convex combination $\frac{1}{t}(b_k + tz) + (1 - \frac{1}{t})q$ with $q = -b_k/(t-1)$—a point near 0, hence interior once $|b_k| \leq \varepsilon\rho/2$—exhibits the nonzero lattice point z in $\mathrm{int}(P)$, contradicting the hypothesis.

The force of Lemma 6 is its *uniformity*: the threshold k_0 depends only on ε and P—not on the field, the basis, or the polytope Q. This matters because the Newton polytope Q_k produced over $\mathbb{F}_p$ depends on the choice of p, while the prime must be chosen larger than $L(Q_k, 0)$; the uniform bound over all admissible Q breaks what would otherwise be a circular dependence, allowing p to be fixed before the $\mathbb{F}_p$-construction is made.

3.5 Reduction modulo p and the upper bound

Reduction modulo p cannot increase the rank of an integer matrix, so the jet coranks satisfy $d^{(p)}_{k,m} \geq d^{(0)}_{k,m}$ and $S_k \leq S_{k,p}$ (equation (6)). Now run the product construction over $\mathbb{F}_p$, so that $\mathrm{ord}_{\mathbf{1}}\, G_k = S_{k,p}$, and pass to the finite ring

$$R_p = \mathbb{F}_p[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]/(z_1^p - 1, \ldots, z_n^p - 1) \;\simeq\; \mathbb{F}_p[y_1, \ldots, y_n]/(y_1^p, \ldots, y_n^p),$$

the isomorphism being Frobenius: $z_i^p - 1 = (z_i - 1)^p$ in characteristic p. Every surviving monomial has total y-degree at most $n(p-1)$, so the maximal ideal satisfies $\mathfrak{m}^{n(p-1)+1} = 0$ (equation (7)). The quotient map $\mathbb{F}_p[z^{\pm 1}] \to R_p$ agrees with expand-at-$\mathbf{1}$-then-truncate, so if $\mathrm{ord}_{\mathbf{1}}\, G_k > n(p-1)$ the image of G_k in R_p is zero. But the monomial classes of R_p form a basis indexed by $(\mathbb{Z}/p\mathbb{Z})^n$, and if $p > L(Q_k, 0)$ no nonzero exponent of G_k is divisible by p—such an exponent would be $pz \in Q_k$ with $z \neq 0$. So the coefficient of the zero residue class is exactly the exposed coefficient of G_k, which is nonzero: the image of G_k in R_p survives. Therefore

$$S_{k,p} = \mathrm{ord}_{\mathbf{1}}\, G_k \leq n(p-1) \qquad \text{whenever } p > L(Q_k, 0). \tag{8}$$

Finally the prime number theorem supplies, for each fixed $\eta > 0$ and all large k, a prime p with $(1 + \eta)kN_k < p < (1 + \eta)^2 kN_k$; by (5) this p exceeds $L(Q_k, 0)$, so (6) and (8) give $S_k/(kN_k) < n(1 + \eta)^2$, and $\eta \downarrow 0$ yields the upper half of the pinch,

$$\limsup_{k \to \infty} \frac{S_k}{kN_k} \leq n. \tag{9}$$

Combining (1) and (9) proves $\mathrm{vol}(P) \leq (n + 1)^n/n!$ (equation (10)), hence the theorem.

Remark 2 (Why characteristic p is necessary). One might hope to shortcut the entire modular apparatus with a characteristic-zero multiplicity bound: *if the origin is an exposed vertex of* $\mathrm{Newt}(G)$ *with nonzero coefficient, then* $\mathrm{ord}_{\mathbf{1}} G \le n \cdot L(\mathrm{Newt}(G), 0)$, which applied to G_k with (5) would give (9) directly. This statement is *false*. There is an explicit 20-term Laurent polynomial $F \in \mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$, supported in the tetrahedron $\mathrm{conv}\{0, (2, -5, 2), (3, -6, 7), (-3, -6, 4)\}$, all of whose nonzero lattice points are primitive—so $L(\mathrm{Newt}(F), 0) = 1$—such that the origin is an exposed vertex of $\mathrm{Newt}(F)$ with coefficient 1, and yet $\mathrm{ord}_{\mathbf{1}} F = 4 > 3 = n \cdot L$. We verified this counterexample by exact rational arithmetic (Section 6.2). The detour through $\mathbb{F}_p$—where the bound $n(p-1)$ holds for the *ring-theoretic* reason $\mathfrak{m}^{n(p-1)+1} = 0$, not for a multiplicity-theoretic one—is thus essential to the argument, not an artifact of its discovery.

Remark 3 (The role of the prime number theorem). Only the sharpness of the constant uses the prime number theorem. Its role is to provide primes of relative gap $o(1)$, so that $n(p-1)/(kN_k) \to n$. If one instead invokes only Bertrand's postulate (a prime in $(x, 2x]$), the identical argument yields $\limsup S_k/(kN_k) \le 2n$ and hence the weaker but fully elementary bound $\mathrm{vol}(K) \le 2^n(n+1)^n/n!$.

4 The analytic proof

We summarize Chapter 8 of [OAI26], following its structure; unqualified labels refer to that chapter. Here K is an arbitrary centered convex body—no polytope reduction is made—and $c_K = (n! \, \mathrm{vol}(K))^{1/n}$.

4.1 The transport potential and lattice Bergman spaces

The barycenter hypothesis enters first, through the existence theorem of Berman and Berndtsson [BB13, Thm. 1.1] (see also the moment-measure framework of [CK15]): there is a smooth strictly convex $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ whose gradient transports $e^{-\varphi(x)}dx$ to Lebesgue measure on K, with $\varphi - h_K$ bounded, $\nabla\varphi$ a diffeomorphism onto $\mathrm{int}(K)$, and $\int e^{-\varphi} = \mathrm{vol}(K)$.

On the complex torus $X = (\mathbb{C}^*)^n$ with $x_i = \log |z_i|^2$ and $d\nu = dx \, d\theta$, let $\mathcal{H}_k$ be the space of holomorphic functions square-integrable against $e^{-k\varphi}d\nu$. Its structure is the bridge between analysis and the lattice (Lemma 2.1): the Laurent monomials z^m with $m \in \mathbb{Z}^n \cap \mathrm{int}(kK)$ form an *orthogonal basis* of $\mathcal{H}_k$—the monomial z^m is integrable precisely when $m/k \in \mathrm{int}(K)$—so $d_k = \dim \mathcal{H}_k = \#(\mathbb{Z}^n \cap \mathrm{int}(kK))$ and $d_k/k^n \to \mathrm{vol}(K)$. In particular *the unique-interior-lattice-point hypothesis is exactly the statement* $\mathcal{H}_1 = \mathbb{C}$; and more generally $\mathcal{H}(b) = \mathbb{C}$ for any weight b with $b - \varphi$ bounded. Lemma 2.2 adds the two limits the construction needs: $k^{-1} \log B_k \to \varphi$ locally uniformly for the Bergman kernels B_k, and the Bergman measures $\mu_k = B_k e^{-k\varphi} d\nu/d_k$ converge in total variation to $\mu = e^{-\varphi}d\nu/\mathrm{vol}(K)$ (a Laplace-method computation).

4.2 The filtration and the lower slope

At $p = \mathbf{1}$, filter $\mathcal{H}_k$ by vanishing order and choose an orthonormal basis (s_a) adapted to the filtration, with orders j_a. Lemma 3.1 is the same codimension count as in Section 3: order $\ge j$ has codimension at most $\binom{n+j-1}{n}$. Truncate the orders at $\lfloor c_K k \rfloor$, setting $q_a = \min\{j_a, \lfloor c_K k \rfloor\}$, and encode them in finite-level potentials and their initial slopes

$$u_t^k = \frac{1}{k}\log\sum_a |s_a|^2 e^{tq_a}, \qquad g_k = \partial_t u_t^k\big|_{t=0} = \frac{\sum_a q_a|s_a|^2}{kB_k}.$$

Averaging against μ_k recovers exactly the normalized aggregate (truncated) order, $\int g_k\, d\mu_k = \frac{1}{kd_k}\sum_a q_a$, and the codimension count gives its sharp lower bound (equation (18)):

$$\liminf_{k\to\infty}\int_X g_k\, d\mu_k \;\geq\; \frac{1}{\operatorname{vol}(K)}\int_0^{c_K}\Big(\operatorname{vol}(K) - \frac{s^n}{n!}\Big)ds = \frac{n}{n+1}c_K.$$

4.3 The ray of potentials

The finite-level data is now welded into a single k-independent object. The functions $U_k(z,\tau) = u_t^k(z)$, $t = \log|\tau|^2$, are plurisubharmonic on $X\times\mathbb{C}^*$ (k^{-1} times the log of a squared norm of a holomorphic vector) and locally uniformly bounded above; their regularized tail envelope defines a plurisubharmonic function Ψ whose slices $\psi_t(z) = \Psi(z, e^{t/2})$ form a ray with

$$\psi_0 = \varphi, \qquad \varphi \leq \psi_t \leq \varphi + c_K t \quad (t\geq 0),$$

convex in t for each z. Its initial velocity $g = \dot\psi_{0^+}$ dominates the finite-level slopes: $\limsup_k g_k \leq g$, and total-variation convergence $\mu_k \to \mu$ transfers the average, giving (Proposition 3.2)

$$\int_X g\, d\mu \;\geq\; \frac{n}{n+1}c_K.$$

4.4 Bergman convexity and the upper slope

Define the normalized partition function along the ray and its logarithm

$$Z(t) = \frac{1}{\operatorname{vol}(K)}\int_X e^{-\psi_t}\, d\nu, \qquad L(t) = -\log Z(t), \qquad Z(0)=1,\ L(0)=0.$$

Here the lattice hypothesis acts a second time, now decisively (Lemma 4.1). Since $\psi_t - \varphi$ is bounded, the weighted space $\mathcal{H}(\psi_t)$ consists of constants only; its Gram matrix is the 1×1 matrix $\operatorname{vol}(K)Z(t)$, so the Bergman kernel of the weight ψ_t *is* the reciprocal partition function, $B_{\psi_t} = e^{L(t)}/\operatorname{vol}(K)$. Berndtsson's positivity theorem [Ber06, Thm. 1.1]—applied with the plurisubharmonic weight $\Psi(z,\tau)$; the change from Euclidean measure to $d\nu$ costs only the pluriharmonic correction $\sum_i \log|z_i|^2$—then makes $(z,\tau)\mapsto \log B_{\psi_t}(z)$ plurisubharmonic, i.e., *L convex*, and dominated convergence identifies $L'_+(0) = \int g\, d\mu$ (equation (24)). Without the hypothesis $\mathcal{H}_1 = \mathbb{C}$ there would be a genuine Gram matrix here and no scalar convexity statement.

The upper bound is a local Schwarz estimate at $\mathbf{1}$ (Lemma 4.2). A unit-norm $s\in\mathcal{H}_k$ vanishing to order j satisfies $|s(\zeta)| \leq C_r e^{kM_r/2}(|\zeta|/r)^j$ in logarithmic coordinates near $\mathbf{1}$; on the ball of radius $re^{-t/2}$ the decay $(|\zeta|/r)^{2j}$ cancels the filtration weight e^{tq_a}, so $\psi_t \leq M_r$ there. That ball has $d\nu$-volume proportional to e^{-nt}—*this is where the constant n is born*—whence $Z(t)\geq c\, e^{-M_r - nt}$ and $L(t)\leq nt + D$; convexity and $L(0)=0$ upgrade this to $L(t)\leq nt$ for all $t\geq 0$ (equation (25)), so $L'_+(0)\leq n$.

The pinch $\frac{n}{n+1}c_K \leq L'_+(0) \leq n$ (equation (2) of the chapter) gives $c_K \leq n+1$, which is Theorem 1.

5 Comparison

5.1 What the proofs share

The shared content is precisely the skeleton of Section 2: the point $\mathbf{1}$; the vanishing-order filtration of a lattice-indexed function space; the codimension count $\binom{n+m-1}{n}$; the resulting lower bound $\frac{n}{n+1}(n!\,\mathrm{vol})^{1/n}$ on the normalized aggregate order, via the same integral; the target upper bound n; and the simplex sharpness computation. The finite-level spaces differ only immaterially ($kP \cap \mathbb{Z}^n$ versus $\mathrm{int}(kK) \cap \mathbb{Z}^n$; both have $\sim \mathrm{vol} \cdot k^n$ points), and [OAI26] truncates the orders at $\lfloor c_K k \rfloor$ so that its potentials have uniformly bounded slope—a need [Init26], which simply sums the finite filtration dimensions, does not have. Since the shared lower half is the elementary half, and it descends from a published argument [Fuj18], the mathematical weight of the comparison falls entirely on the upper halves.

5.2 Where they diverge

	Algebraic proof [Init26]	**Analytic proof [OAI26]**
Generality of the core argument	Centered rational polytopes; general bodies by Hausdorff approximation (Lemma 4).	Arbitrary centered convex bodies, directly.
Barycenter hypothesis enters via	The first lattice moment: $b_k = U_k/(kN_k) \to 0$, feeding the radius bound (Lemma 6).	Existence of the Monge–Ampère transport potential φ [BB13].
Lattice hypothesis enters via	The convex-combination contradiction in Lemma 6: a long lattice multiple in $kN_kP - U_k$ would force a nonzero interior lattice point.	$\mathcal{H}_1 = \mathbb{C}$, hence $\mathcal{H}(\psi_t) = \mathbb{C}$: the Gram matrix is 1×1 and the Bergman kernel equals the reciprocal partition function (Lemma 4.1).
Packaging of the vanishing orders	Multiplicative: the adapted basis is multiplied into one Laurent polynomial G_k with $\mathrm{ord}_{\mathbf{1}}\, G_k = S_k^F$ and an exposed Newton vertex at 0.	Variational: the orders become exponential weights e^{tq_a} in log-sum-exp potentials, regularized into a plurisubharmonic ray ψ_t with slope $L'_+(0)$.
Rigidity mechanism (upper half)	The exposed coefficient survives in $R_p \simeq \mathbb{F}_p[y]/(y_1^p, \ldots, y_n^p)$, whose maximal ideal is nilpotent: $S_{k,p} \le n(p-1)$.	Berndtsson positivity makes L convex; a Schwarz estimate on a shrinking ball gives $L(t) \le nt$; convexity converts this into $L'_+(0) \le n$.
Origin of the constant n	The socle degree $n(p-1)$ of the truncated group algebra: n directions of capacity $p-1$ each.	The volume $\asymp e^{-nt}$ of a complex n-ball of radius $\asymp e^{-t/2}$.
Asymptotic devices	Lattice Riemann sums; a prime of relative gap $o(1)$ via the prime number theorem.	Laplace asymptotics of Bergman kernels; total-variation convergence $\mu_k \to \mu$; upper-semicontinuous envelopes; reverse Fatou.
Deepest external inputs	The prime number theorem (and only for the sharp constant; Remark 3).	Existence and regularity of the transport potential [BB13]; Berndtsson's positivity theorem [Ber06].

	Algebraic proof [Init26]	**Analytic proof [OAI26]**
Equality cases	Not determined.	Not determined [OAI26, after Thm. 1.1]; the classification conjectured in [NP14, Conj. 1.1] has since been established by Liu [Liu26] (Section 7).

5.3 Are the proofs genuinely different?

Yes. The clearest evidence is functional: track where each hypothesis acts. In the analytic proof the barycenter hypothesis is consumed at the very start, to build the ambient potential φ; in the algebraic proof it is consumed near the end, as the elementary statement that a lattice average tends to zero. In the analytic proof the lattice hypothesis collapses a Gram matrix to a scalar so that a positivity theorem applies; in the algebraic proof it excludes long lattice rays from a Newton polytope so that a residue class stays clean. Neither role has a counterpart on the other side: nothing in [Init26] plays the part of a plurisubharmonic ray or a partition function, and nothing in [OAI26] plays the part of a product of basis elements, an exposed vertex, or a positive characteristic.

There is nevertheless a structural rhyme worth recording. In both proofs the unique-interior-lattice-point hypothesis protects a one-dimensional object that is the sole survivor of a collapse, and the protected object is what forbids excessive vanishing. Analytically, the survivor is the constant function: $\mathcal{H}(\psi_t) = \mathbb{C}$ turns the Bergman kernel into the reciprocal partition function, and the Schwarz estimate says a function positive on a shrinking ball of volume e^{-nt} cannot decay faster than e^{-nt}. Algebraically, the survivor is the residue class of 0 in R_p: no other exponent of G_k can alias onto it once p exceeds the lattice radius, and nilpotency says a nonzero element of R_p cannot vanish past degree $n(p-1)$. One may read the two upper bounds as continuous and characteristic-p manifestations of one principle—*a protected nonzero one-dimensional trace is incompatible with vanishing beyond the capacity of an n-dimensional local model*—with capacity measured by ball volume in one case and socle degree in the other. We stress that this is an interpretation, an analogy between the roles played by disjoint pieces of machinery; it is not a dictionary, and we found no step-by-step translation between the arguments.

6 Verification

6.1 The audit of the manuscript

We verified [Init26] line by line: every lemma, every displayed equation, and every constant. Our verdict is that the proof is correct. The manuscript is self-contained; its external inputs are classical, are named at their point of use, and consist of the prime number theorem (in the form of prime gaps of relative size $o(1)$), continuity of volume and barycenter under Hausdorff convergence of convex bodies, and standard convexity facts. The audit of the argument was completed before we consulted [OAI26]; the two texts corroborate each other on the statement and on every shared constant.

The points we probed hardest, and why each is sound:

1. *Negative exponents.* The entire jet apparatus must survive Laurent (not just polynomial)

exponents. It does: the generalized binomial coefficients $\binom{u_i}{\alpha_i}$ are integers for all $u_i \in \mathbb{Z}$, and the expansion map E is injective on Laurent polynomials (clear denominators by a monomial, which E sends to a unit).

2. *No cancellation at the exposed vertex, over every field.* The argument that the coefficient of z^{U_k} in F_k is the product of the pivot coefficients uses only the strict ℓ-minimality of pivots and is characteristic-free; nothing about orthogonality or positivity is smuggled in.

3. *The apparent circularity in the choice of* p. The bound (8) needs $p > L(Q_k, 0)$, but Q_k is built over $\mathbb{F}_p$ and so depends on p. The resolution is the uniformity of Lemma 6: the radius bound holds for *every* polytope contained in $kN_kP - U_k$, with a threshold $k_0(\varepsilon, P)$ independent of field, basis, and polytope. The order of quantifiers in the manuscript is correct, and the uniformity is genuine, not asserted.

4. *The two routes into* R_p. The upper bound uses that the canonical quotient $\mathbb{F}_p[z^{\pm 1}] \to R_p$ agrees with expansion at $\mathbf{1}$ followed by truncation. The manuscript's one-line proof (both are algebra maps sending $z_i \mapsto 1 + y_i$, and such a map out of the Laurent ring is determined by the invertible images of the z_i) is complete.

5. *Sign conventions and bookkeeping.* The Riemann-sum comparisons in (1) and (4), the direction of the rank inequality under reduction mod p, and the η-accounting in the final limit were checked to be stated with the correct signs and orders of limits.

6.2 Computational corroboration

We machine-checked the mechanism of [Init26] at finite level with exact rational arithmetic (fraction-based Gaussian elimination over $\mathbb{Q}$; elimination over $\mathbb{F}_p$), on the extremal body for $n = 2$: the triangle $P = 3\Delta_2 - (1,1) = \{x \geq -1,\ y \geq -1,\ x + y \leq 1\}$ with $\mathrm{vol}(P) = \frac{9}{2} = (n+1)^n/n!$. For $k = 1, \ldots, 4$ we computed the jet matrices with entries $\binom{u_1}{\alpha_1}\binom{u_2}{\alpha_2}$, their coranks $d^{(0)}_{k,m}$ and $d^{(p)}_{k,m}$ for every m, the aggregates S_k and $S_{k,p}$, the worst-case radius $\Lambda_k = L(kN_kP, 0)$ (here $U_k = 0$ by symmetry), and the least prime $p > \Lambda_k$. Every inequality asserted by the proof held at every level: $S_k \geq \sum_m (N_k - J_m)_+$, and $d^{(p)}_{k,m} \geq d^{(0)}_{k,m}$ for all m, and $S_k \leq S_{k,p} \leq n(p-1)$.

k	N_k	S_k	$\sum_m (N_k - J_m)_+$	Λ_k	p	$S_{k,p}$	$n(p-1)$
1	10	20	20	10	11	20	20
2	28	112	112	56	59	112	116
3	55	330	330	165	167	330	332
4	91	728	728	364	367	728	732

Two exact identities in this data are worth recording. First, $S_k/(kN_k) = 2 = n$ *exactly* at every computed level, and the jet lower bound is attained with equality ($S_k = \sum_m (N_k - J_m)_+$): on the extremal body the two halves of the pinch (2.1) meet at n already at finite k, not merely in the limit. Second, at $k = 1$ the whole chain is simultaneously tight: with the minimal admissible prime $p = 11$ one has $S_1 = S_{1,11} = 20 = n(p-1)$. The constants of the proof are honest—there is no slack anywhere in the extremal case.

The same exact-arithmetic methodology verified the counterexample of Remark 2: complete enumeration of the lattice points of the tetrahedron, primitivity of all nonzero points, vanishing of all moments of F of order ≤ 3, non-vanishing at order 4, and an independent truncated Taylor expansion at $\mathbf{1}$ confirming $\operatorname{ord}_{\mathbf{1}} F = 4$.

6.3 The status of the analytic proof

We read Chapter 8 of [OAI26] in full and checked its structure—the statements of its lemmas, the flow of implications, the use of its hypotheses, and all constants—against both the chapter's own internal logic and the correspondence of Section 5. We did not referee its analytic infrastructure (the Laplace asymptotics, envelope regularity, and the adaptation of [Ber06]) at the same depth as [Init26], and this note should not be cited as an independent verification of Chapter 8. We note that the chapter's two deepest inputs are published, refereed theorems [BB13, Ber06], and that OpenAI released a Lean 4 certificate for the result [OAIL26], which machine-checks a formal statement of the theorem modulo the usual caveat of matching the formal statement to the informal one. Nothing we found in Chapter 8 is in tension with [Init26] at any point of contact; on the contrary, the two proofs agree on every joint constant, which is mutual corroboration of a strong kind, since the agreements are forced by disjoint mechanisms.

7 Discussion

The value of a second proof. For a conjecture open for six decades, a second proof by disjoint methods has independent scientific weight. The theorem now rests on two pillars that share only elementary counting: a reader who distrusts pluripotential theory can follow [Init26]; a reader who distrusts analytic number theory can follow [OAI26]. Each proof de-risks the other's deepest dependencies.

Audit surface. The two proofs sit at opposite ends of the verifiability spectrum. The algebraic proof's prerequisites are lattice-point asymptotics, linear algebra of flags, Newton polytopes, finite fields, and the prime number theorem; we were able to referee it completely, and to machine-check its finite-level mechanism exactly, within this note's preparation. The analytic proof compresses its difficulty into two cited theorems and a nontrivial amount of pluripotential bookkeeping; auditing it to the same depth is a substantially larger undertaking, for which its Lean certificate [OAIL26] is the compensating asset.

A transferable mechanism. The algebraic upper half isolates a chain that appears to be new in convex geometry:

$$\begin{aligned}\text{jet filtration} \;\longrightarrow\; \text{two-flag adapted product} \;\longrightarrow\; \text{exposed Newton vertex}\\ \longrightarrow\; \text{survival mod } p \;\longrightarrow\; \text{nilpotency bound,}\end{aligned}$$

an obstruction in the spirit of the polynomial method [CLP17] glued to convex-geometric data by the radius bound. Finite-level jet counts and saturation arguments closely related to the filtration side of this chain appear in Fano geometry in work of Zhang [Zha22, Zha25]; see [Liu26, Rem. 1.5] for a dictionary of such parallels. To our knowledge the positive-characteristic steps have no counterpart on that side. The ingredients are portable: what is needed is a graded space of Laurent polynomials, asymptotic control of its exponent set, and a lattice-exclusion hypothesis. Whether the chain proves other Fujita-type slope bounds—for instance in toric or graded-Ehrhart settings [RR24]—is a natural question we do not pursue here.

Finite-level exactness. The computations of Section 6.2 show that on the extremal body the algebraic proof's inequalities are exact at every finite level, not only asymptotically. The analytic proof is intrinsically limiting: its objects (ψ_t, μ, $L'_+(0)$) exist only after envelopes and total-variation limits. Both proofs as written, however, discard this rigidity in their final limits (Riemann sums, $o(1)$ radius estimates, prime gaps, and Hausdorff approximation on one side; Laplace asymptotics, envelopes, and $T \to \infty$ convexity on the other), and neither yields the equality classification directly.

The equality case. As this note was being completed, Liu [Liu26] established the classification conjectured in [NP14, Conj. 1.1]: the equality bodies are exactly the unimodular images of $(n+1)\Delta_n - \mathbf{1}$. The proof works within the analytic framework, using the transport potential and the weighted spaces of [OAI26] directly: the volume hypothesis forces the analogue of the partition function L of Section 4, based at an arbitrary point of the torus, to be exactly linear; a radial degeneration and the equality case of the Prékopa–Leindler inequality then force the body to be a simplex, and a critical-lattice theorem for the extremal simplex, proved via Hajós's theorem on cube tilings, pins the lattice. What remains open is precisely the effective side suggested by the finite-level exactness above: stability estimates—quantitative bounds forcing a near-extremal body close to the extremal simplex—and an algebraic proof of the equality case. Since [Liu26] rests on the analytic pillar, the latter would extend to the full conjecture the independence of methods that the inequality now enjoys.

Necessity of the modular detour. Remark 2 shows the algebraic proof's characteristic-p step is not a stylistic choice: the characteristic-zero bound that would replace it is false, by an explicit and machine-verified counterexample. The truncated group algebra is doing work that no multiplicity bound at an exposed vertex can do.

Conclusion

Theorem 1 now has two proofs: one analytic, one algebraic. They agree on the easy half—the same Fujita-style filtration count at the same point, with the same constants—and solve the hard half with machinery so disjoint that no step of one argument maps to a step of the other. We have verified the algebraic proof completely, corroborated its mechanism by exact computation, and found it correct and self-contained; its comparison with Chapter 8 of [OAI26] locates precisely where the two routes coincide, where they rhyme, and where they part. The equality classification of [NP14, Conj. 1.1], open when both proofs were written, has since been established by Liu [Liu26] within the analytic framework; stability estimates, and an algebraic route to the equality case, are the natural remaining frontier.

8 AI-assisted work

This work — research, development, verification, writing, and review — was produced with extensive use of AI systems, including large language models.

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