

Spectral Edges and Large- q Asymptotics in the Sachdev–Ye–Kitaev Model

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Abstract

For every fixed even interaction order $q \geq 4$, we determine the upper and lower spectral edges of the Sachdev–Ye–Kitaev Hamiltonian. In the standardized normalization

$$H_{N,q} = \binom{N}{q}^{-1/2} \sum_{|A|=q} J_A \Psi_A,$$

we prove, as $N \rightarrow \infty$ through even integers,

$$\frac{\lambda_{\max}(H_{N,q})}{\sqrt{N}} \rightarrow E_q, \quad \frac{\lambda_{\min}(H_{N,q})}{\sqrt{N}} \rightarrow -E_q \quad \text{almost surely,}$$

and consequently $\|H_{N,q}\|/\sqrt{N} \rightarrow E_q$. The constant is characterized by the unique normalized positive-Laplace solution g_q of the zero-temperature Schwinger–Dyson equation and by a strictly concave variational principle:

$$E_q = \frac{2^{q/2+1}}{\sqrt{q}} \int_0^\infty g_q(t)^q dt = 2^{q/2} \sqrt{q} \max_{\sigma \in \mathcal{X}_q} \mathcal{F}_q(\sigma).$$

The proof extends the finite-bath interpolation and cavity method of He from $q = 4$ to arbitrary fixed even interaction order, relying on the operator-algebraic transfer results of He (arXiv:2607.18998, Sections 4–5). We isolate the arity-neutral form of those results, verify their model-dependent hypotheses for fixed q , and prove the remaining Majorana, locality, cavity, zero-temperature, and concentration estimates directly.

We then analyze the fixed- q variational formula and prove the sequential asymptotic expansion

$$E_q = \frac{\sqrt{2}}{q} \left(1 - \frac{12 - \pi^2}{6q} + O(q^{-2}) \right), \quad q \rightarrow \infty, \quad q \text{ even.}$$

The large- q argument is an exact primal–dual comparison based on a rational positive-Laplace trial profile and a Gamma–Laplace dual certificate; once the fixed- q variational formula is available it is a self-contained deterministic argument and uses none of the operator-algebraic input. The order of limits is essential: $N \rightarrow \infty$ is taken first for each fixed q ; no joint $q = q(N)$ or double-scaled limit is asserted.

Keywords. Sachdev–Ye–Kitaev model; spectral edge; Majorana fermions; Schwinger–Dyson equation; operator algebras; large- q asymptotics.

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1 Introduction and main results

1.1 Background and contribution

The Sachdev–Ye–Kitaev (SYK) Hamiltonian [1, 2, 3] is a random q -body operator on a Hilbert space of dimension $2^{N/2}$, built from $\binom{N}{q}$ independent Gaussian couplings. In the standardized normalization used below, the normalized second spectral moment remains of order one, while the extreme eigenvalues grow on the larger scale $\sqrt{N}$. Equivalently, in the extensive normalization the edge is linear in N , and its limiting coefficient is the zero-temperature energy density. The edge therefore probes a regime that is not determined by convergence of the global density of states.

We use the convention $\exp(+\beta\mathcal{H}_{N,q})$, so the zero-temperature pressure selects the upper edge; $e_q > 0$ denotes its limiting extensive density, and the conventional ground-state energy density of $\mathcal{H}_{N,q}$ is $-e_q$ whenever the two limiting edges are symmetric.

Rigorous results on the SYK model include the limiting spectral distribution and related bulk statistics [5], order-sharp lower bounds on the extreme energy [6], and high-temperature annealed and quenched free-energy limits [8]. In a recent preprint, He proves the exact quartic spectral edge and an all-temperature annealed-pressure formula by a finite-bath interpolation and a local cavity-kernel identity [4]. That work indicates that the method should extend to fixed $q \geq 6$, but does not carry out the extension. To our knowledge, the extension to every fixed even interaction order and the rigorous large- q correction proved below are new.

We first give a complete sextic proof, with all arity-dependent constants displayed explicitly. We determine the normalization, prove existence and uniqueness of the limiting zero-temperature Schwinger–Dyson object, identify the edge through both an integral and a variational principle, and establish expected and almost-sure convergence of both spectral edges. The proof is then formulated for every fixed even $q \geq 4$ in Theorem 12.1. Finally, Section 13 analyzes the resulting deterministic constants and proves their leading large- q value and first correction.

Several features of the fixed-arity argument change with the interaction order. For $q = 6$, the Hermitian phase is $i^3 = -i$, the positive-Lehmann mass of the fifth-power source is $1/16$, the one-site cavity matching factor is $2^6 = 64$, and the zero-temperature decay available from the moment bounds is $g_6(t) = O(t^{-1/3})$. More generally, the corresponding quantities are $i^{q/2}$, 2^{2-q} , and 2^q . The proof below tracks these factors explicitly, including the fermionic crossing signs and all finite- N normalization constants.

The large- q analysis is the paper’s second result. Anschuetz, Chen, Kiani, and King proved the order-sharp lower scaling $\Omega(\sqrt{N}/q)$ [6], while the large- N physics saddle predicted the constant $\sqrt{2}$ and a specific $1/q$ correction. We prove that prediction for the fixed-arity edge constants, identifying the constant in that scaling and its first correction:

$$\frac{qE_q}{\sqrt{2}} = 1 - \frac{12 - \pi^2}{6q} + O(q^{-2}).$$

The proof is primal–dual and deterministic once the fixed- q variational formula is known. It also yields explicit finite- q integral certificates that sandwich $qE_q/\sqrt{2}$; the window has width $O(q^{-2})$ and is wide at small q .

The zero-temperature Schwinger–Dyson prediction for the edge is therefore a theorem beyond the quartic model, and its leading large- q correction is proved rather than assumed. We do not obtain bounds uniform in q , and nothing here applies to $q = q(N)$ or to the double-scaled limit.

1.2 Scope and external inputs

The fixed-order proof uses the arity-neutral operator-algebraic transfer results of He [4, Sections 4–5], together with the specialized external results summarized in Appendix A. The hypotheses needed from

those results are restated, and the new model-dependent hypotheses are verified in Proposition 6.9, Theorem 6.10, and their fixed- q analogues. The paper does not reproduce the full Groh–Raynaud, Connes–cocycle, Takesaki, and direct-integral machinery. All arity-dependent Majorana algebra, combinatorics, interpolation coefficients, locality estimates, cavity signs, zero-temperature arguments, concentration bounds, and large- q calculations are proved here. Once the fixed- q variational characterization is established, the large- q section is a self-contained deterministic real-analysis argument and uses none of the imported operator-algebraic machinery. All citations to He refer to arXiv:2607.18998v1, whose numbering is used throughout the proof-dependency summary.

1.3 Probability space, Majoranas, and Hamiltonian

Let

$$\Omega = \mathbb{R}^{\binom{N}{6}}, \quad \mathbb{P} = \bigotimes_{A \in \binom{[N]}{6}} \mathcal{N}(0, 1),$$

and let J_A be the coordinate variables. For each even $N \geq 6$, take an irreducible representation of the complex Clifford algebra generated by self-adjoint Majoranas $\chi_1, \dots, \chi_N$ satisfying

$$\chi_i \chi_j + \chi_j \chi_i = \delta_{ij} I, \quad \mathbf{q}_i := \sqrt{2} \chi_i, \quad \mathbf{q}_i^2 = I.$$

All spectral quantities below are independent of the chosen irreducible realization, because the complex Clifford algebra in even dimension has a unique irreducible representation up to unitary equivalence. For $A = \{i_1 < \dots < i_6\} \subset [N]$, set

$$\Psi_A := i^3 \mathbf{q}_{i_1} \cdots \mathbf{q}_{i_6} = -i \mathbf{q}_{i_1} \cdots \mathbf{q}_{i_6}. \quad (1)$$

Then Ψ_A is a Hermitian unitary and

$$\Psi_A \Psi_B = (-1)^{|A \cap B|} \Psi_B \Psi_A, \quad \mathbf{q}_i \Psi_A \mathbf{q}_i = (-1)^{1_{\{i \in A\}}} \Psi_A. \quad (2)$$

The standardized sextic SYK Hamiltonian is

$$H_{N,6} := \binom{N}{6}^{-1/2} \sum_{A \in \binom{[N]}{6}} J_A \Psi_A. \quad (3)$$

We write Tr for the ordinary matrix trace and $\text{tr} := 2^{-N/2} \text{Tr}$ for the normalized trace.

Define the extensive Hamiltonian and the finite- N arity factor by

$$\mathcal{H}_{N,6} := 2^{-3} \sqrt{\frac{5!}{N^5}} \sum_{|A|=6} J_A \Psi_A, \quad \alpha_{N,6} := \prod_{j=1}^5 \left(1 - \frac{j}{N}\right). \quad (4)$$

A direct calculation gives the conversion

$$\mathcal{H}_{N,6} = \sqrt{\frac{\alpha_{N,6} N}{6 \cdot 2^6}} H_{N,6} = \sqrt{\frac{\alpha_{N,6} N}{384}} H_{N,6}. \quad (5)$$

Thus the standardized edge scale is $\sqrt{N}$ and the extensive edge scale is N , and the natural normalization is $a_N = \sqrt{N}$. The finite- N alternative $a_N^{\text{ex}} = \sqrt{N/\alpha_{N,6}}$ removes the factor $\alpha_{N,6}$ from (5) and is asymptotic to $\sqrt{N}$.

1.4 The sextic zero-temperature Schwinger–Dyson object

For a probability measure ρ on $[0, \infty)$ define

$$g_\rho(t) := \frac{1}{2} \int_0^\infty e^{-Et} \rho(dE), \quad R_\rho(v) := \int_0^\infty \frac{v}{v^2 + E^2} \rho(dE).$$

A normalized sextic zero-temperature Schwinger–Dyson solution is a probability measure ρ_6 such that, with $g_6 = g_{\rho_6}$,

$$\int_0^\infty E^2 \rho_6(dE) = \frac{1}{16}, \quad g_6^5 \in L^1(0, \infty), \quad (6)$$

$$R_6(v) := R_{\rho_6}(v) = \frac{1}{v + S_6(v)}, \quad S_6(v) := 2 \int_0^\infty \sin(vt) g_6(t)^5 dt, \quad v > 0. \quad (7)$$

Sections 9–10 prove existence and uniqueness in this class.

Define

$$E_6 := \frac{16}{\sqrt{6}} \int_0^\infty g_6(t)^6 dt. \quad (8)$$

The integral is finite because Theorem 9.2 gives $g_6(t)^6 \leq Ct^{-2}$ for $t \geq 1$.

Theorem 1.1 (Sextic spectral edge). *For the coupled family (3), as $N \rightarrow \infty$ through even integers,*

$$\frac{\lambda_{\max}(H_{N,6})}{\sqrt{N}} \longrightarrow E_6 \quad \text{almost surely.} \quad (9)$$

For every finite N and every realization, the spectrum is exactly symmetric, $\lambda_{\min}(H_{N,6}) = -\lambda_{\max}(H_{N,6})$. Hence on the same probability-one event,

$$\frac{\lambda_{\min}(H_{N,6})}{\sqrt{N}} \longrightarrow -E_6, \quad \frac{\|H_{N,6}\|}{\sqrt{N}} \longrightarrow E_6. \quad (10)$$

The constant E_6 is equivalently the variational value

$$E_6 = 8\sqrt{6} \max_{\sigma \in \mathcal{X}_6} \mathcal{F}_6(\sigma), \quad (11)$$

where

$$\mathcal{X}_6 := \left\{ \sigma(t) = \int_0^\infty e^{-Et} \nu(dE) : \nu \geq 0 \text{ finite, } \sigma \in L^1 \cap L^{6/5} \right\}, \quad (12)$$

$$S_\sigma(v) := 2 \int_0^\infty \sin(vt) \sigma(t) dt, \quad (13)$$

$$\mathcal{F}_6(\sigma) := \frac{1}{2\pi} \int_0^\infty \log \left(1 + \frac{S_\sigma(v)}{v} \right) dv - \frac{5}{6} \int_0^\infty \sigma(t)^{6/5} dt. \quad (14)$$

The unique maximizer is $\sigma_6 = g_6^5$.

Remark 1.2 (Exact sextic spectral symmetry). *The lower edge is stronger than a distributional consequence of $J \mapsto -J$. Lemma 11.3 constructs an antiunitary that fixes every Majorana generator and reverses the sextic phase i^3 . It anticommutes with $H_{N,6}$, so the spectrum is symmetric samplewise at every finite N .*

Theorem 1.3 (Large- q asymptotics, summary). *For the fixed-arity constants in Theorem 12.1,*

$$E_q = \frac{\sqrt{2}}{q} \left(1 - \frac{12 - \pi^2}{6q} + O(q^{-2}) \right) \quad (q \rightarrow \infty, \text{ } q \text{ even}). \quad (15)$$

Equivalently, on the simultaneous probability-one event of Corollary 12.7,

$$\lim_{\substack{q \rightarrow \infty \\ q \text{ even}}} \left(\lim_{N \rightarrow \infty} \frac{q \lambda_{\max}(H_{N,q})}{\sqrt{2N}} \right) = 1.$$

1.5 Proof strategy and organization

The proof has three stages. First, at each fixed inverse temperature, we approximate the proposed self-energy by a finite quadratic Majorana bath and interpolate between that bath and the SYK interaction. Gaussian integration by parts expresses the derivative through a nonnegative Bregman remainder. The main issue is to show that the remainder vanishes as $N \rightarrow \infty$.

Second, we delete interactions meeting more than one marked site, prove a forced-incidence N^{-1} locality gain, and apply the modular and de Finetti transfer results from [4, Sections 4–5] on each fixed finite coordinate family. A slow deterministic diagonal then allows the number of marked sites to grow. Restoring a distinguished site by a summable cavity expansion produces the quadratic source $\theta\beta^2 G^{q-1}$; collision terms are $O(N^{-1})$, and the residual odd Majorana blocks produce precisely the Pfaffian crossing sign. Uniqueness of the positive-Lehmann fixed point makes the limiting kernel deterministic and forces the interpolation remainder to vanish.

Third, we pass to dimensionful Euclidean time, obtain uniform tails, identify the unique zero-temperature Schwinger–Dyson solution and the pressure slope, and derive the variational formula. The entropy sandwich transfers the pressure slope to the expected top eigenvalue, while Gaussian concentration and Borel–Cantelli yield almost-sure convergence. Section 2 records the quartic dependency structure; Sections 3–11 give the complete sextic proof; Section 12 proves the fixed-even- q extension; and Section 13 proves the large- q expansion by explicit primal and dual certificates. Appendix A summarizes the use of computational tools, the nonessential consistency checks, the proof-critical external inputs, and the scope of the conclusions.

2 The quartic proof and the extension strategy

2.1 The quartic theorem and conventions

For $q = 4$, He [4] uses

$$\Psi_A = i^2 \prod_{i \in A}^{\nearrow} \mathbf{q}_i = - \prod_{i \in A}^{\nearrow} \mathbf{q}_i, \quad H_{N,4} = \binom{N}{4}^{-1/2} \sum_{|A|=4} J_A \Psi_A.$$

The extensive Hamiltonian is

$$\mathcal{H}_{N,4} = 2^{-2} \sqrt{\frac{3!}{N^3}} \sum_{|A|=4} J_A \Psi_A, \quad \alpha_{N,4} = \frac{(N-1)(N-2)(N-3)}{N^3},$$

so

$$\mathcal{H}_{N,4} = \frac{\sqrt{\alpha_{N,4} N}}{8} H_{N,4}. \quad (16)$$

On the unit thermal circle, put

$$k_x(\tau) := \frac{\cosh(x(1/2 - \tau))}{2 \cosh(x/2)}, \quad G_\mu(\tau) := \int_0^\infty k_x(\tau) \mu(dx). \quad (17)$$

A normalized positive Lehmann kernel has μ a probability measure, hence $G_\mu(0+) = 1/2$.

For a finite positive measure ρ , let $\bar{\rho}$ be the image under $x \mapsto x^2$ and define

$$S_\rho(z) := \int_0^\infty \frac{\bar{\rho}(dy)}{z+y}, \quad F_\rho(z) := \frac{1}{z(1+S_\rho(z))}. \quad (18)$$

He's Dyson lemma states that F_ρ is the Stieltjes transform of a unique probability measure; pulling that measure back under $y = x^2$ defines $D(\rho)$ and the kernel map $D(G_\rho) := G_{D(\rho)}$. Its output second moment equals the mass $\rho([0, \infty))$. The determinant ratio is

$$\mathcal{D}(G_\rho) := \prod_{n \geq 0} \left(1 + \frac{s_\rho(\omega_n)}{\omega_n} \right), \quad s_\rho(\omega) := \omega \int_0^\infty \frac{\rho(dx)}{\omega^2 + x^2}, \quad \omega_n = (2n+1)\pi. \quad (19)$$

The finite-temperature quartic solution is the unique normalized positive Lehmann kernel satisfying

$$G_{\beta,4} = D(\beta^2 G_{\beta,4}^3), \quad \int x^2 \mu_{\beta,4}(dx) = \frac{\beta^2}{4}. \quad (20)$$

Its pressure is

$$p_{\text{SD},4}(\beta) = \log \mathcal{D}(\beta^2 G_{\beta,4}^3) - \frac{3\beta^2}{8} \int_0^1 G_{\beta,4}(u)^4 du. \quad (21)$$

The zero-temperature kernel g_4 is the unique positive Laplace solution with probability spectral measure, second moment $1/4$, and $g_4^3 \in L^1$. He's theorem is

$$\frac{\lambda_{\max}(H_{N,4})}{\sqrt{N}} \longrightarrow \kappa_4 := 4 \int_0^\infty g_4(t)^4 dt \quad \text{almost surely,} \quad (22)$$

with the reported numerical value

$$\kappa_4 = 0.32504215806675932177147800181384 \dots$$

2.2 Proof decomposition

The table records what each stage of the quartic argument supplies.

Source result	Role in the proof
Lemma 2.1	Positive closure $G_\mu^3 = G_{\Gamma_3(\mu)}$, mass $1/4$, and weak continuity.
Lemmas 2.2–2.4	The Dyson reciprocal preserves positive Lehmann form, fixes the output second moment, is continuous on bounded-mass sets, and is anti-monotone in the pairing $\int (G_1 - G_2)(\Sigma_1 - \Sigma_2)$.
Proposition 2.5 and Theorem 2.6	Schauder existence, strict-monotonicity uniqueness, and parameter continuity of $G = D(\rho + aG^3)$.
Lemma 3.1 and Proposition 3.2	Every finite atomic positive source is realized by a finite quadratic Majorana bath; the target source $\beta^2 G_\beta^3$ is approximated with fixed mass $\beta^2/4$.
Propositions 3.3 and 3.6	Interpolation endpoints and derivative $P'_M = (\beta^2 \alpha_{N,4}/8) \int K_4 - (1/2) \int \Sigma_M G$.
Corollary 3.7	Pressure comparison with remainder $K_4 - 4h^3 G + 3h^4$.
Propositions 4.4–4.5	Deleting quartets with two marked sites costs $O_\beta(k^2/N)$; four independent flips differ from four distinct flips by $O(1/k)$.
Lemmas 4.1–4.9 and Theorems 4.10–4.11	Gaussian derivative estimates and support incidence force one N^{-1} factor in unequal-time marked-field commutators, uniformly on compact real-time sets.
Lemmas 4.14–4.19	Relative modular spectral tightness, equal ultraproduct supports, commuting coordinate algebras, and modular invariance on a common faithful corner.

Source result	Role in the proof
Lemmas 5.2–5.9 and Proposition 5.10	Projective exchangeability and Takesaki conditional expectations produce one product directing law and conditional factorization of the required contour words.
Corollary 5.12 and Lemmas 6.2–6.5	Disjoint triple assignments factor into $(2G)^3$; collisions are $O(N^{-1})$; anticommuting residual triple blocks give the Pfaffian crossing sign.
Propositions 6.6, Corollary 6.8, Lemma 6.9	The restored site is a quadratic bath with source $\theta\beta^2 G_\lambda^3$, and exchangeability gives the conditional fixed point.
Propositions 6.10–6.11 and Corollary 6.12	Fixed-point stability plus $(x-r)^2(x^2+2rx+3r^2)$ makes the integrated interpolation remainder vanish.
Theorem 1.2	The annealed pressure converges to (21) at every fixed β .
Proposition 7.2 and Theorems 7.3–7.4	$g_\beta(t) \leq Ct^{-1/2}$, compactness, recovery of the second moment, and uniqueness of the zero-temperature equation.
Theorem 7.6, Proposition 7.7, Corollary 7.8	Midpoint-sum convergence and a strictly concave/convex variational argument give $\lim p_{\text{SD},4}(\beta)/\beta = (1/2) \int g_4^4$.
Proposition 8.1 and entropy sandwich	Gaussian Lipschitz concentration, expected-edge convergence, and Borel–Cantelli yield (22).

2.3 Where quarticity enters

The table records each place where the quartic argument uses $q = 4$, and what replaces it at $q = 6$.

Quartic occurrence	Why it is used	Sextic form
$\Psi_A = -\mathbf{q}_{i_1} \cdots \mathbf{q}_{i_4}$	Makes the even monomial Hermitian and unitary.	A sixfold product is anti-Hermitian; $\Psi_A = -i\mathbf{q}_{i_1} \cdots \mathbf{q}_{i_6}$.
G^3 has positive Lehmann mass $1/4$	Gives a positive source and fixes the spectral second moment.	Fifth-power closure with mass $1/16$; Lemma 4.1.
Strict monotonicity of x^3	Selects a unique Dyson fixed point.	x^5 is strictly increasing on $[0, \infty)$; the same argument applies.
$c_4^2 = 3/(8N^3)$ and factor 16	Fix interpolation and cavity coefficients.	$c_{N,6}^2 = 15/(8N^5)$; cavity factor 64.
Quartets/triples in locality counts	A forced intersection leaves $O(N^2)$ triples against variance $O(N^{-3})$.	Sextets/five-sets leave $O(N^4)$ labels against variance $O(N^{-5})$, again $O(N^{-1})$.
Four independent flips	Replaces with-replacement overlap by f_4 .	Six flips produce f_6, f_4, f_2, f_0 with the coefficients in Lemma 6.2.
Odd residual cubic blocks	Two disjoint residual blocks anticommute and crossings give -1 .	Residual fivefold blocks are also odd, square to $-I$, and anticommute; Lemma 7.3.
Remainder $x^4 - 4r^3x + 3r^4$	Nonnegative Bregman divergence for x^4 .	$x^6 - 6r^5x + 5r^6 \geq 0$, factored in (99).
Tail $g(t) \lesssim t^{-1/2}$	Supplies L^3 and L^4 tails.	$g(t) \lesssim t^{-1/3}$, supplying L^5 and L^6 tails.
Penalty exponent $4/3$	Convex dual of the cubic self-energy.	$6/5$; dilation scales as $a^{7/5}$.

Quartic occurrence	Why it is used	Sextic form
Triple-index anticommuting family in concentration	Bounds the Gaussian gradient by $(N - 3)^{-1/2}$.	Fix a five-set; the gradient bound becomes $(N - 5)^{-1/2}$.
Modular/de Finetti machinery	Uses only fixed support, exchangeability, locality, and bounded local baths.	No quartic algebra remains after the sextic locality hypotheses are verified.

2.4 Dimensional and scaling check

The normalized trace orthogonality of distinct Clifford monomials gives

$$\mathbb{E} \operatorname{tr} H_{N,q}^2 = 1, \quad \mathbb{E} \operatorname{tr} \mathcal{H}_{N,q}^2 = \frac{\alpha_{N,q} N}{q 2^q}.$$

Thus $H_{N,q}$ has bulk spectral scale one but an extreme scale $\sqrt{N}$, while $\mathcal{H}_{N,q}$ has variance of order N and an extreme scale N . The unit-circle Green function has normalization $G(0+) = 1/2$, and the representing mass of the finite-temperature source $\beta^2 G^{q-1}$ and the Dyson output second moment are both $2^{2-q} \beta^2$. Under $t = \beta\tau$ and $E = x/\beta$, the second moment becomes the β -independent number 2^{2-q} . Every factor in (21) is therefore order one per site at fixed β , and the zero-temperature pressure is order β .

3 Sextic Majorana algebra and normalization

Lemma 3.1 (Correct Hermitian phase). *For every even $q = 2m$ and every ordered q -set A , the operator*

$$\Psi_A = i^m \prod_{i \in A}^{\nearrow} \mathbf{q}_i$$

is a Hermitian unitary. If A, B both have even cardinality, then

$$\Psi_A \Psi_B = (-1)^{|A \cap B|} \Psi_B \Psi_A.$$

For $q = 6$ this gives (1).

Proof. Let $Q_A = \prod_{i \in A}^{\nearrow} \mathbf{q}_i$. Reversing the product requires $q(q-1)/2$ transpositions, hence

$$Q_A^* = (-1)^{q(q-1)/2} Q_A = (-1)^m Q_A, \quad Q_A^2 = (-1)^m I.$$

Also $\overline{i^m} = (-1)^m i^m$ and $(i^m)^2 = (-1)^m$. Therefore

$$(i^m Q_A)^* = (-1)^m i^m (-1)^m Q_A = i^m Q_A, \quad (i^m Q_A)^2 = (-1)^m (-1)^m I = I.$$

To interchange Q_A and Q_B , every pair of distinct Majoranas contributes one transposition. The number is $|A||B| - |A \cap B|$. Since $|A||B|$ is even,

$$Q_A Q_B = (-1)^{|A \cap B|} Q_B Q_A.$$

The scalar phases commute and cancel. □

Lemma 3.2 (Standardized/extensive conversion). *Let*

$$c_{N,q} := 2^{-q/2} \sqrt{\frac{(q-1)!}{N^{q-1}}}, \quad \alpha_{N,q} := \frac{(N-1) \cdots (N-q+1)}{N^{q-1}}.$$

Then

$$c_{N,q} \sum_{|A|=q} J_A \Psi_A = \sqrt{\frac{\alpha_{N,q} N}{q 2^q}} \binom{N}{q}^{-1/2} \sum_{|A|=q} J_A \Psi_A.$$

In particular, (5) holds and $c_{N,6}^2 = 15/(8N^5)$.

Proof. Since

$$\binom{N}{q} = \frac{N(N-1) \cdots (N-q+1)}{q!} = \frac{N^q \alpha_{N,q}}{q!},$$

the square of the conversion factor is

$$c_{N,q}^2 \binom{N}{q} = 2^{-q} \frac{(q-1)!}{N^{q-1}} \frac{N^q \alpha_{N,q}}{q!} = \frac{\alpha_{N,q} N}{q 2^q}.$$

For $q = 6$, $2^{-6} 5! = 120/64 = 15/8$. □

4 Positive Lehmann closure and the finite-temperature sextic equation

4.1 The fifth-power closure

For a probability measure μ on $[0, \infty)$ and a bounded Borel function f , define a positive measure $\Gamma_5(\mu)$ by

$$\begin{aligned} \int f(r) \Gamma_5(\mu)(dr) &:= \int_{[0, \infty)^5} \mu(dx_1) \cdots \mu(dx_5) \sum_{\varepsilon_2, \dots, \varepsilon_5 \in \{\pm 1\}} W_\varepsilon(x) \\ &\quad \times f\left(x_1 + \sum_{j=2}^5 \varepsilon_j x_j\right), \end{aligned} \quad (23)$$

where

$$W_\varepsilon(x) := \frac{\cosh\left((x_1 + \sum_{j=2}^5 \varepsilon_j x_j)/2\right)}{256 \prod_{j=1}^5 \cosh(x_j/2)}. \quad (24)$$

Lemma 4.1 (Positive fifth-power closure). *For every probability measure μ ,*

$$G_\mu(\tau)^5 = G_{\Gamma_5(\mu)}(\tau), \quad \Gamma_5(\mu)([0, \infty)) = \frac{1}{16}. \quad (25)$$

The map $\mu \mapsto \Gamma_5(\mu)$ is continuous for weak convergence of probability measures.

Proof. The elementary identity

$$\prod_{j=1}^5 \cosh(a_j) = \frac{1}{16} \sum_{\varepsilon_2, \dots, \varepsilon_5 = \pm 1} \cosh\left(a_1 + \sum_{j=2}^5 \varepsilon_j a_j\right) \quad (26)$$

follows by expanding each $\cosh(a_j) = (e^{a_j} + e^{-a_j})/2$ and pairing a sign vector with its negative. Insert $a_j = x_j(1/2 - \tau)$ and divide by $2^5 \prod_j \cosh(x_j/2)$. For

$$s = x_1 + \sum_{j=2}^5 \varepsilon_j x_j,$$

the term on the right is

$$\frac{\cosh(s(1/2 - \tau))}{512 \prod_j \cosh(x_j/2)} = W_\varepsilon(x) k_{|s|}(\tau),$$

because $\cosh(s/2) = \cosh(|s|/2)$. Integration against $\mu^{\otimes 5}$ proves the kernel identity.

Putting $\tau = 0$ in (26), or applying the same identity with $a_j = x_j/2$, gives

$$\sum_{\varepsilon} W_\varepsilon(x) = \frac{16 \prod_j \cosh(x_j/2)}{256 \prod_j \cosh(x_j/2)} = \frac{1}{16}.$$

Each weight is nonnegative and bounded by $1/16$. For bounded continuous f , the integrand in (23) is bounded and continuous. Weak convergence of product measures proves continuity. $\square$

4.2 Dyson map, anti-monotonicity, and the sextic fixed point

We use the Dyson map (18)–(19). The following statement is He's Lemmas 2.2–2.4, whose proof is independent of the nonlinear exponent.

Lemma 4.2 (Dyson positivity and sign). *Let ρ be a finite positive measure. There is a unique probability measure $D(\rho)$ whose squared-variable Stieltjes transform is F_ρ in (18). Its second moment is $\rho([0, \infty))$. On every class of uniformly bounded source mass, D and $\mathcal{D}$ are weakly continuous and*

$$1 \leq \mathcal{D}(G_\rho) \leq \exp\{\rho([0, \infty))/8\}.$$

If $G_j = D(\Sigma_j)$, with Σ_j positive Lehmann kernels, then

$$\int_0^1 (G_1 - G_2)(\Sigma_1 - \Sigma_2) du = -2 \sum_{n \geq 0} \frac{(s_{1,n} - s_{2,n})^2}{(\omega_n + s_{1,n})(\omega_n + s_{2,n})} \leq 0. \quad (27)$$

Proof. For completeness, the positivity representation is obtained on $\mathbb{C} \oplus L^2(\bar{\rho})$ from the closed operator

$$B(\zeta, f)(y) = \zeta - \sqrt{y}f(y).$$

The spectral measure of B^*B at $(1, 0)$ has Stieltjes transform $[z(1 + S_\rho(z))]^{-1}$. The form norm of $B(1, 0)$ gives the first moment in the squared variable, hence the second moment after pullback. Weak continuity follows from convergence of Stieltjes transforms and the common second-moment bound. The determinant bound uses $\log(1 + x) \leq x$ and $\sum_{n \geq 0} \omega_n^{-2} = 1/8$.

For (27), write $\hat{\Sigma}_j(n) = i s_{j,n}$ and $\hat{G}_j(n) = i g_{j,n}$ for $n \geq 0$. The reciprocal formula gives $g_{j,n} = (\omega_n + s_{j,n})^{-1}$. Parseval and pairing n with $-n - 1$ yield

$$2 \sum_{n \geq 0} (g_{1,n} - g_{2,n})(s_{1,n} - s_{2,n}),$$

and substituting the reciprocal difference gives (27). $\square$

Proposition 4.3 (Affine fifth-power fixed point). *Let ρ be a finite positive measure and $a \geq 0$. There is a unique normalized positive Lehmann kernel G satisfying*

$$G = D(G_\rho + aG^5). \quad (28)$$

Its representing probability measure has second moment

$$\rho([0, \infty)) + \frac{a}{16}. \quad (29)$$

The solution is continuous, in weak measure topology and in every finite $L^r(0, 1)$, under weak variation of ρ with bounded mass and ordinary variation of a .

Proof. Let m_2 be the quantity in (29), and let K_{m_2} be the convex set of probability measures with second moment at most m_2 . It is tight and weakly closed, hence weakly compact by Prokhorov's theorem [12]. Lemma 4.1 and Lemma 4.2 show that

$$\mu \mapsto D(\rho + a\Gamma_5(\mu))$$

is a continuous self-map of K_{m_2} ; every output has second moment exactly m_2 . Schauder–Tychonoff gives a fixed point.

For uniqueness, let G_1, G_2 be fixed points and set $\Sigma_j = G_\rho + aG_j^5$. If $a = 0$, the map is constant. If $a > 0$, (27) gives

$$0 \geq a \int_0^1 (G_1 - G_2)(G_1^5 - G_2^5) du \geq 0.$$

The integrand is nonnegative and vanishes only when $G_1 = G_2$. Continuity of the kernels and injectivity of the Lehmann representation give equality of the measures.

For parameter continuity, uniform second moments give tightness. Every subsequential weak limit satisfies the limiting fixed-point equation by the continuity lemmas, and uniqueness identifies it. On the real thermal interval, $0 \leq k_x(u) \leq 1/2$; dominated convergence gives pointwise and every finite L^r convergence. $\square$

Theorem 4.4 (Finite-temperature sextic Schwinger–Dyson solution). *For every $\beta \geq 0$ there is a unique normalized positive Lehmann kernel $G_{\beta,6} = G_{\mu_{\beta,6}}$ satisfying*

$$G_{\beta,6} = D(\Sigma_{\beta,6}), \quad \Sigma_{\beta,6} = \beta^2 G_{\beta,6}^5. \quad (30)$$

For $\beta > 0$,

$$\int_0^\infty x^2 \mu_{\beta,6}(dx) = \frac{\beta^2}{16}. \quad (31)$$

The map $\beta \mapsto \mu_{\beta,6}$ is weakly continuous.

Proof. Apply Proposition 4.3 with $\rho = 0$ and $a = \beta^2$. $\square$

Definition 4.5 (Sextic Schwinger–Dyson pressure). *Set*

$$p_{\text{SD},6}(\beta) := \log \mathcal{D}(\Sigma_{\beta,6}) - \frac{5\beta^2}{12} \int_0^1 G_{\beta,6}(u)^6 du. \quad (32)$$

The coefficient $5/12$ is derived in Proposition 5.1.

5 Finite quadratic bath and sextic interpolation

5.1 Atomic approximation

By Lemma 4.1, $\Sigma_{\beta,6}$ has a positive Lehmann representing measure

$$\nu_{\beta,6} := \beta^2 \Gamma_5(\mu_{\beta,6}), \quad \nu_{\beta,6}([0, \infty)) = \frac{\beta^2}{16}.$$

Choose finite atomic measures ν_M of the same mass, converging weakly to $\nu_{\beta,6}$, and write

$$\Sigma_M(u) = \sum_{\alpha=1}^M s_{\alpha} k_{x_{\alpha}}(u), \quad s_{\alpha} = v_{\alpha}^2, \quad \sum_{\alpha} s_{\alpha} = \frac{\beta^2}{16}. \quad (33)$$

Then, by dominated convergence and Lemma 4.2,

$$\Sigma_M \rightarrow \Sigma_{\beta,6} \text{ in every finite } L^r, \quad \int \Sigma_M^{6/5} \rightarrow \int \Sigma_{\beta,6}^{6/5}, \quad \log \mathcal{D}(\Sigma_M) \rightarrow \log \mathcal{D}(\Sigma_{\beta,6}). \quad (34)$$

The common pointwise bound is

$$0 \leq \Sigma_M(u) \leq \frac{\beta^2}{32}, \quad (35)$$

because $0 \leq k_x \leq 1/2$.

Attach to every physical χ_i and every atom a pair of bath Majoranas $\eta_{i\alpha}, \zeta_{i\alpha}$. The one-site quadratic term

$$ix_{\alpha} \eta_{i\alpha} \zeta_{i\alpha} + iv_{\alpha} \chi_i \eta_{i\alpha}$$

realizes the source by the Schur-complement identity of [4, Lemma 3.1]. This identity is purely quadratic and has no dependence on q .

5.2 Interpolation and overlaps

Let

$$\mathcal{H}_{M,\theta}(J) = -\sqrt{\theta} \beta \mathcal{H}_{N,6}(J) + \sum_{i=1}^N \sum_{\alpha=1}^M \left(ix_{\alpha} \eta_{i\alpha} \zeta_{i\alpha} + i\sqrt{1-\theta} v_{\alpha} \chi_i \eta_{i\alpha} \right). \quad (36)$$

Normalize the annealed partition function by the free bath factors exactly as in [4, (3.9)], and put $P_M(\theta) = N^{-1} \log Z_M(\theta)$. The endpoint identities are

$$P_M(0) = \log \mathcal{D}(\Sigma_M), \quad P_M(1) = p_{N,6}(\beta) := \frac{1}{N} \log \mathbb{E} \operatorname{tr} e^{\beta \mathcal{H}_{N,6}}. \quad (37)$$

For a sign set $S \subset [N]$, let $\alpha_S = \prod_{i \in S} \operatorname{Ad}(\mathbf{q}_i)$ and define the normalized two-state overlap

$$f_{S,M,\theta}(u) := \frac{\mathbb{E} \operatorname{Tr} (e^{(1-u)A_{\theta,J}} e^{u\alpha_S(A_{\theta,J})})}{\mathbb{E} \operatorname{Tr} e^{A_{\theta,J}}}, \quad A_{\theta,J} := -\mathcal{H}_{M,\theta}(J). \quad (38)$$

Petz positivity and Schatten Holder give

$$0 \leq f_{S,M,\theta}(u) \leq 1. \quad (39)$$

Let f_j be the average over j distinct sites and set

$$G_{M,\theta} := \frac{1}{2} f_1, \quad K_{6,M,\theta} := \frac{1}{64} f_6. \quad (40)$$

The factors are forced by $\mathbf{q}_i = \sqrt{2} \chi_i$: one raw Majorana contributes $2^{-1/2}$ and six raw Majoranas contribute 2^{-3} at each side of the thermal word, hence 2^{-6} .

Proposition 5.1 (Sextic interpolation derivative). *Put*

$$\mathbf{a}_{N,6} := \beta^2 \alpha_{N,6}.$$

For $0 < \theta < 1$,

$$P'_M(\theta) = \frac{\mathbf{a}_{N,6}}{12} \int_0^1 K_{6,M,\theta}(u) \, du - \frac{1}{2} \int_0^1 \Sigma_M(u) G_{M,\theta}(u) \, du. \quad (41)$$

If

$$h_M(u) := \left(\frac{\Sigma_M(u)}{\mathbf{a}_{N,6}} \right)^{1/5}, \quad (42)$$

then

$$\begin{aligned} P'_M(\theta) &= -\frac{5\mathbf{a}_{N,6}}{12} \int_0^1 h_M(u)^6 \, du \\ &\quad + \frac{\mathbf{a}_{N,6}}{12} \int_0^1 [K_{6,M,\theta} - 6h_M^5 G_{M,\theta} + 5h_M^6] (u) \, du. \end{aligned} \quad (43)$$

Both identities extend integrably to $[0, 1]$.

Proof. Differentiate the sextic part of $A_{\theta,J}$ and integrate by parts in each J_A . Duhamel's formula gives one thermal word with two copies of Ψ_A . Since $\Psi_A^2 = I$ and conjugation by Ψ_A equals the sign action α_A , its normalized value is the six-site overlap. The coefficient is

$$\begin{aligned} \frac{\beta^2 c_{N,6}^2}{2N} \binom{N}{6} 64 &= \frac{\beta^2}{2N} \frac{15}{8N^5} \frac{N^6 \alpha_{N,6}}{720} 64 \\ &= \frac{\beta^2 \alpha_{N,6}}{12}. \end{aligned}$$

This proves the first term of (41).

Differentiating $\sqrt{1-\theta}$ in the bath hybridization and applying the two-dimensional boundary-value identity [4, Lemma 3.5] gives

$$-\frac{1}{2} \int_0^1 \Sigma_M(u) G_{M,\theta}(u) \, du.$$

That identity uses only the CAR of one physical Majorana and one bath pair, so its hypotheses are unchanged.

Substitute $\Sigma_M = \mathbf{a}_{N,6} h_M^5$ and add and subtract $5\mathbf{a}_{N,6} h_M^6/12$. This gives (43). The bounds $K_6 \leq 1/64$, $G \leq 1/2$, and (35) make the derivative uniformly bounded for fixed M, β , while finite-dimensional endpoint continuity gives absolute continuity on $[0, 1]$. $\square$

Corollary 5.2 (Finite-bath pressure comparison). *With*

$$\mathcal{R}_{M,N}(u, \theta) := K_{6,M,\theta}(u) - 6h_M(u)^5 G_{M,\theta}(u) + 5h_M(u)^6, \quad (44)$$

one has

$$\begin{aligned} p_{N,6}(\beta) &= \log \mathcal{D}(\Sigma_M) - \frac{5\mathbf{a}_{N,6}}{12} \int_0^1 h_M(u)^6 \, du \\ &\quad + \frac{\mathbf{a}_{N,6}}{12} \int_0^1 \int_0^1 \mathcal{R}_{M,N}(u, \theta) \, du \, d\theta. \end{aligned} \quad (45)$$

Proof. Integrate (43) and use (37). $\square$

6 Sextic locality, deletion, and the arity-neutral modular limit

This section supplies the hypotheses needed by the operator-algebraic part of the quartic proof. We state the sextic estimates in the notation of Sections 4–5 of [4]; the proofs below expose the only arity-dependent counting.

6.1 Deleted sextic sign system

Let Λ be a bulk set of cardinality n and let $\mathbf{M}$ be a disjoint marked set of fixed cardinality k . After deleting every sextic interaction containing at least two marked sites, write

$$X_i = \gamma \sum_{T \in \binom{\Lambda}{5}} g_{i,T} \Psi_{\{i\} \cup T}, \quad K_0 = \eta \sum_{E \in \binom{\Lambda}{6}} J_E \Psi_E, \quad (46)$$

where all displayed Gaussian variables are independent and

$$\gamma^2 \leq \alpha_1 n^{-5}, \quad \eta^2 \leq \alpha_0 n^{-5}. \quad (47)$$

A site-local bath is a self-adjoint operator $B = B^*$, written as a sum $B = \sum_{i,\ell} b_{i\ell}$, such that $b_{i\ell}$ is even, acts only at site i and its attached bath, commutes with every word whose physical support omits i , does not enlarge physical support under commutation, and satisfies

$$\sup_{n,\theta,i} \sum_{\ell} \|b_{i\ell}\| \leq \Lambda_B < \infty. \quad (48)$$

At a marked site there may also be a bounded self-adjoint even local hybridization $Z_i = Z_i^*$, with $\|Z_i\| \leq \Lambda_Z$, satisfying

$$[Z_i, Z_j] = [Z_i, X_j] = 0 \quad (i \neq j), \quad [Z_i, K_0] = 0, \quad (49)$$

and commuting with bath terms based away from i . Writing $\alpha_i = \text{Ad}(\mathbf{q}_i)$, assume in addition that the bath and hybridization obey the sign-action identities

$$\alpha_i(B) = B, \quad \alpha_i(Z_i) = -Z_i, \quad \alpha_j(Z_i) = Z_i \quad (j \neq i). \quad (50)$$

Put $\tilde{X}_i = X_i + Z_i$ and

$$A_\sigma = K_0 + \sum_{i \in \mathbf{M}} \sigma_i \tilde{X}_i + B, \quad \sigma \in \{\pm 1\}^{\mathbf{M}}. \quad (51)$$

The support and parity relations give $\alpha_i(K_0) = K_0$, $\alpha_i(X_i) = -X_i$, and $\alpha_i(X_j) = X_j$ for $j \neq i$. Together with (50), conjugation by $\mathbf{q}_i$ changes exactly the i th sign in A_σ . All sector partition functions are therefore equal samplewise.

The physical interpolation model fits this framework. If $N = n + k$, then its Gaussian coefficient square is $\theta\beta^2 15/(8N^5)$, which is bounded by $C_{k,\beta} n^{-5}$; the free atomic bath satisfies (48)–(49) and (50) at fixed cutoff.

6.2 Deletion and six-flip combinatorics

Proposition 6.1 (Sextic multi-mark deletion). *Let $L \subset [N]$ have cardinality k . Delete every sextet meeting L in at least two sites. For every normalized annealed contour word of fixed heat complexity, total heat length one, bounded total real-time length, and fixed atomic bath, the original and deleted expectations*

differ by at most a word-dependent constant times

$$\delta_{k,6}(N, \theta, \beta) := \theta \beta^2 \frac{15}{8N^5} \sum_{j=2}^6 \binom{k}{j} \binom{N-k}{6-j} \quad (52)$$

$$\leq \frac{5\beta^2}{128} \frac{k(k-1)}{N}. \quad (53)$$

Starting from this deleted model, let F be any additional set of cardinality r and delete every further sextet meeting $L \cup F$ in at least two sites. The additional comparison error is bounded by a word-dependent constant times

$$\delta_{k,r}^{\text{add}} \leq \frac{5\beta^2}{64N} \left(kr + \binom{r}{2} \right). \quad (54)$$

The bounds are uniform in the labels, sign sectors, and $\theta \in [0, 1]$.

Proof. Restore the deleted Gaussian coordinates by a variance parameter and differentiate the quotient of contour numerator by partition function. Gaussian heat flow gives one coefficient square per restored direction. The trace-word derivative estimate of [4, Lemma 4.3] applies because every inserted Ψ_A is a contraction and all sector generators are unitarily conjugate. Thus the comparison is bounded by a fixed contour constant times the sum of the restored coefficient squares, which is exactly (52).

Every sextet counted in (52) contains a marked pair. A union bound over pairs gives

$$\sum_{j=2}^6 \binom{k}{j} \binom{N-k}{6-j} \leq \binom{k}{2} \binom{N-2}{4}.$$

Using $\binom{N-2}{4} \leq N^4/24$ gives

$$\frac{15}{8N^5} \binom{k}{2} \binom{N-2}{4} \leq \frac{15}{8N^5} \frac{k(k-1)}{2} \frac{N^4}{24} = \frac{5}{128} \frac{k(k-1)}{N}.$$

For the additional deletion, every newly removed sextet contains either one pair with one endpoint in L and one in $F \setminus L$, or a pair contained in $F \setminus L$. There are at most $kr + \binom{r}{2}$ choices for such a pair and at most $\binom{N-2}{4}$ completions. Multiplication by $15\beta^2/(8N^5)$ proves (54). $\square$

Let f_j denote the permutation-invariant overlap for j distinct marked coordinates. Let a_6 be the overlap obtained after six independent uniform marked-coordinate flips, with repeated flips canceled modulo two.

Lemma 6.2 (Six-flip decomposition). *For $k \geq 6$,*

$$a_6 = A_{6,k} f_6 + A_{4,k} f_4 + A_{2,k} f_2 + A_{0,k}, \quad (55)$$

where

$$A_{6,k} = \frac{(k)_6}{k^6}, \quad (56)$$

$$A_{4,k} = \frac{15(k)_5 + 20(k)_4}{k^6}, \quad (57)$$

$$A_{2,k} = \frac{45(k)_4 + 75(k)_3 + 16(k)_2}{k^6}, \quad (58)$$

$$A_{0,k} = \frac{15(k)_3 + 15(k)_2 + k}{k^6}. \quad (59)$$

The coefficients are nonnegative and sum to one. Since every overlap lies in $[0, 1]$,

$$|a_6 - f_6| \leq 1 - \frac{(k)_6}{k^6} \leq \frac{15}{k}. \quad (60)$$

Proof. A pattern of equalities among the six ordered draws is a set partition of six positions. The surviving flips are precisely the blocks of odd size. The patterns with six odd blocks contribute $(k)_6$.

Four odd blocks arise from multiplicities $2 + 1 + 1 + 1 + 1$ and $3 + 1 + 1 + 1$, contributing respectively $15(k)_5$ and $20(k)_4$. Two odd blocks arise from

$$2 + 2 + 1 + 1, \quad 4 + 1 + 1, \quad 3 + 2 + 1, \quad 5 + 1, \quad 3 + 3,$$

whose contributions are $45(k)_4$, $15(k)_3$, $60(k)_3$, $6(k)_2$, and $10(k)_2$. Zero odd blocks arise from $2 + 2 + 2$, $4 + 2$, and 6 , contributing $15(k)_3$, $15(k)_2$, and k . This gives (56)–(59); summing all set partitions gives k^6 .

Equation (55) writes $a_6 = A_{6,k}f_6 + (1 - A_{6,k})q$ for some $q \in [0, 1]$. The first inequality in (60) follows. The probability that six independent draws contain a collision is at most the union bound over their $\binom{6}{2} = 15$ pairs, which proves the second inequality. $\square$

6.3 The forced-incidence locality estimate

For a sector-propagator word L define the right annealed GNS seminorm

$$\|F\|_L^2 := \frac{\mathbb{E} \operatorname{Tr}(e^{A_+} L F F^* L^*)}{\mathbb{E} \operatorname{Tr} e^{A_+}}. \quad (61)$$

For distinct marked sites a, b , put

$$B_{p,p'} := [(\operatorname{ad} A_+)^p X_a, (\operatorname{ad} A_+)^{p'} X_b], \quad r = p + p'.$$

Lemma 6.3 (Sextic block-exposure estimate). *For every $T_0 < \infty$ there is $C_{T_0} = C(\alpha_0, \alpha_1, \Lambda_B, T_0)$ such that, uniformly in n, p, p' , in the sign sector, and in sector-propagator words of total absolute time at most T_0 ,*

$$\|B_{p,p'}\|_L^2 \leq \frac{C_{T_0}^{r+1} (r+1)^{2r+4} (1+k)^{2r}}{n}. \quad (62)$$

The same estimate holds for the full fields $\tilde{X}_a, \tilde{X}_b$, after enlarging the constant by a function of Λ_Z .

Proof. Expand a nested commutator in construction order and discard every label assignment for which an intermediate commutator vanishes. Before any construction position, the physical support is contained in the union of the initial field, the terminal field, and at most r sextic labels. Hence its cardinality is at most

$$h_r := 6r + 12. \quad (63)$$

If a bulk sextet is admissible, even disjoint locality forces it to meet this support. Therefore

$$\#\{E \in \binom{\Lambda}{6} : E \cap S \neq \emptyset\} \leq \sum_{u=1}^6 \binom{|S|}{u} \binom{n-|S|}{6-u} \leq C(r+1)n^5, \quad (64)$$

$$\#\{T \in \binom{\Lambda}{5} : T \cap S \neq \emptyset\} \leq \sum_{u=1}^5 \binom{|S|}{u} \binom{n-|S|}{5-u} \leq C(r+1)n^4. \quad (65)$$

The estimates remain valid when $h_r > n$, after increasing the numerical constant.

Square two ordered branches and apply repeated Gaussian integration by parts in the partial-matching form of [4, Lemma 4.2]. Order Gaussian blocks by their first exterior occurrence. Follow the inner construction on the first branch from X_a to the terminal commutator with X_b . Select the first b -family adjoint if one occurs, and otherwise select the terminal X_b . Immediately before this selected block, site b is absent: bulk labels lie in Λ ; another marked family carries a different marked site; and a bath

letter based at b would commute with a word omitting b and hence could not be admissible. A nonzero commutator therefore forces the selected five-set to meet the earlier support. By (65), its label sum is at most $C(r+1)n^4$. The partial-matching formula assigns one coefficient square to this block, whether it is paired or differentiates the terminal trace word. By (47), its total conditional mass is at most

$$C(r+1)n^4\alpha_1n^{-5} \leq \frac{C(r+1)}{n}. \quad (66)$$

Every other bulk block has conditional mass at most $C(r+1)n^5\alpha_0n^{-5} \leq C(r+1)$. Every other marked block has mass at most $\binom{n}{5}\alpha_1n^{-5} \leq C$. At a bath occurrence, the total operator-norm weight of admissible site-local letters is at most $\Lambda_B|S| \leq C\Lambda_B(r+1)$. Later equalities of labels and later admissibility constraints can only reduce these sums.

There are at most $r+2$ exterior Gaussian occurrences on one branch and hence at most $2r+4$ after squaring. If s pairs occur in the partial matching, their number is at most $d^{2s}/s!$, where $d \leq 2r+4$. Each singleton derivative of the terminal heat/real-time trace word costs at most $1+2T_0$ by [4, Lemma 4.1]. Summing the partial matchings and the finite choices of branch orientation and constituent type gives

$$\frac{1}{n} \sum_{s \geq 0} \frac{d^{2s}}{s!} \{C(1+2T_0)(r+1)\}^{d-s} \{C(1+\Lambda_B)(r+1)\}^{2r+4-d} (1+k)^{2r}.$$

Since $d \leq 2r+4$, the exponential generated by the s -sum is bounded by C^{r+1} . This is (62).

For $\tilde{X}_i = X_i + Z_i$, an admissible branch beginning with Z_i remains in the even local i -algebra until an internal random X_i occurs. A nonzero branch connecting distinct sites therefore still contains the forced random b -block above. The Z letters do not enlarge physical support and have bounded norm. They only enlarge the constant. $\square$

Technical remark 6.4 (Analytic continuation and the derivative count). Lemma 6.3 is a squared-GNS estimate and by itself does *not* yield a complex-time neighborhood. Its formal total-degree majorant contains

$$\frac{(r+1)^{2r+4}}{r!} u^r,$$

whose consecutive-term ratio is asymptotic to $e^2 u r$ and therefore diverges for every $u > 0$. That estimate does not imply a positive complex-time radius. Instead, as in the quartic proof, we differentiate the entire squared-commutator function before applying partial-matching integration by parts. This leaves only $r+4$ exterior Gaussian occurrences at total derivative order r and produces the summable majorant $(r+1)^{r+4} u^r / r!$. Lemmas 6.5–6.6 prove that sharper sextic estimate explicitly.

For a sector-propagator word L , fixed distinct marked sites a, b , and a sign sector σ , define

$$\mathfrak{f}_{n,\sigma,L}^{a,b}(z, w) := -\frac{\mathbb{E} \operatorname{Tr}(e^{A_\sigma} L[Y_{a,\sigma}(z), Y_{b,\sigma}(w)]^2 L^*)}{\mathbb{E} \operatorname{Tr} e^{A_\sigma}}, \quad Y_{i,\sigma}(z) := e^{-izA_\sigma} \tilde{X}_{i,\sigma} e^{izA_\sigma}. \quad (67)$$

For each finite n this is entire on $\mathbb{C}^2$. Differentiation under the disorder expectation is justified on compact subsets because every derivative is bounded by a polynomial in the finitely many Gaussian coordinates times an exponentially integrable linear-Gaussian majorant. For real z, w ,

$$\mathfrak{f}_{n,\sigma,L}^{a,b}(z, w) = \frac{\mathbb{E} \operatorname{Tr}(e^{A_\sigma} L[Y_{a,\sigma}(z), Y_{b,\sigma}(w)][Y_{a,\sigma}(z), Y_{b,\sigma}(w)]^* L^*)}{\mathbb{E} \operatorname{Tr} e^{A_\sigma}} \geq 0. \quad (68)$$

Lemma 6.5 (Origin analytic estimate at the correct derivative order). *Fix positive integers k, m and $T < \infty$. There are constants $C_0 < \infty$ and $\delta_0 > 0$, depending only on the fixed Gaussian-scale and*

bath bounds and on k, m, T , such that, uniformly in n , the sign sector, the distinct sites a, b , and every sector-propagator word L with at most m factors and total absolute time at most T ,

$$\sup_{|z|+|w|\leq\delta_0} |\mathfrak{f}_{n,\sigma,L}^{a,b}(z, w)| \leq \frac{C_0}{n}. \quad (69)$$

More precisely, for $p, p' \geq 0$ and $r = p + p'$,

$$\left| \partial_z^p \partial_w^{p'} \mathfrak{f}_{n,\sigma,L}^{a,b}(0, 0) \right| \leq \frac{C_0^{r+1} (r+1)^{r+4} (1+k)^r}{n}. \quad (70)$$

Proof. The covariance identity

$$[Y_{a,\sigma}(z), Y_{b,\sigma}(w)] = e^{-iw \operatorname{ad} A_\sigma} [Y_{a,\sigma}(z-w), \tilde{X}_{b,\sigma}] \quad (71)$$

puts every mixed derivative of total order r into an outer-inner nested-commutator form. This point is where the derivative count differs from the squared GNS estimate in Lemma 6.3: after differentiating the *whole holomorphic square* in (67), the total number d of exterior Gaussian occurrences is at most

$$d \leq r + 4, \quad (72)$$

not $2r + 4$. The four undifferentiated occurrences are the two endpoint fields in each of the two algebraic commutators; each complex derivative adds at most one Gaussian Hamiltonian letter. Bounded local Z - and bath letters do not count as Gaussian occurrences.

Expand each derivative in construction order and discard every assignment for which an intermediate commutator vanishes. On the first nonzero algebraic commutator branch, follow the construction from the a -field to the terminal b -field. If the endpoint is the random part X_b , select the first internal b -family letter, or the terminal X_b if no such internal letter occurs. If the endpoint is the local part Z_b , a nonzero branch must contain an internal random X_b , because Z_b commutes with every word whose physical support omits b ; select the first such internal X_b . Immediately before the selected block, site b is absent. Its five-set must therefore meet a support of size at most $6r + 12$, so (65) gives at most $C(r+1)n^4$ labels. The partial-matching integration-by-parts formula assigns this block one coefficient square, whether it is paired or is a singleton differentiating the terminal trace word. Thus its conditional mass is at most

$$C(r+1)n^4\alpha_1n^{-5} \leq \frac{C(r+1)}{n}.$$

The same argument covers a branch beginning with Z_a : until the first internal X_a it remains inside the even local a -algebra and cannot connect to the b -field.

Every other bulk block has conditional mass at most $C(r+1)n^5\alpha_0n^{-5}$, every other marked block has bounded mass, and every deterministic site-local branch has total norm weight at most $C(1+\Lambda_B+\Lambda_Z)(r+1)$. If s Gaussian pairs are removed, their number is at most $d^{2s}/s!$. The singleton derivatives of the alternating terminal trace word cost at most a fixed power of $1+2T$, by the trace-word derivative estimate [4, Lemma 4.1]. Consequently the contribution of all partial matchings and all unlabelled branch patterns is bounded by

$$\frac{1}{n} \sum_{s \geq 0} \frac{d^{2s}}{s!} \{C_T(r+1)\}^{d-s} \{C_T(r+1)\}^{r+4-d} (1+k)^r.$$

Using $d \leq r + 4$ and

$$\sum_{s \geq 0} \frac{d^{2s}}{s!} \{C_T(r+1)\}^{-s} \leq \exp \left\{ \frac{d^2}{C_T(r+1)} \right\} \leq C^r,$$

and absorbing the multinomial choices of where the p and p' derivatives fall, gives (70).

Finally, for $|z| + |w| = u$,

$$\begin{aligned} |\mathfrak{f}_{n,\sigma,L}^{a,b}(z,w)| &\leq \frac{C_0}{n} \sum_{r \geq 0} C_0^r (1+k)^r (r+1)^{r+4} \sum_{p+p'=r} \frac{|z|^p |w|^{p'}}{p! p'!} \\ &= \frac{C_0}{n} \sum_{r \geq 0} \frac{(r+1)^{r+4}}{r!} \{C_0(1+k)u\}^r. \end{aligned}$$

The ratio of consecutive terms tends to $eC_0(1+k)u$. Choosing, for example, $\delta_0 < [2eC_0(1+k)]^{-1}$ makes this series uniformly convergent and proves (69). $\square$

Lemma 6.6 (Uniform analytic bound about real centers). *Fix positive integers k, m and $T, R < \infty$. There are constants $C_R < \infty$ and $\delta_R > 0$, independent of n , the sign sector, a, b, L , and the real center $|x|, |y| \leq R$, such that*

$$\sup_{|\zeta|+|\eta| \leq \delta_R} |\mathfrak{f}_{n,\sigma,L}^{a,b}(x+\zeta, y+\eta)| \leq C_R. \quad (73)$$

For $p + p' = r$ one may take the derivative bound

$$\left| \partial_\zeta^p \partial_\eta^{p'} \mathfrak{f}_{n,\sigma,L}^{a,b}(x+\zeta, y+\eta) \Big|_{\zeta=\eta=0} \right| \leq C_R^{r+1} (r+1)^{r+4} (1+k)^r. \quad (74)$$

Proof. Since every phase is generated by the same sector generator, $Y_i(x+\zeta) = e^{-ixA_\sigma} Y_i(\zeta) e^{ixA_\sigma}$. Expanding at (x, y) therefore produces the same nested words as at the origin, interspersed with sector-propagator histories whose total absolute time is at most $8R + 2T$. Again $d \leq r + 4$. Repeat the block expansion in the preceding proof but do not select a forced-incidence block. Every Gaussian block then has conditional mass at most $C_R(r+1)$, and the same partial-matching sum gives (74). Dividing by $p! p'!$ and summing by total degree gives the convergent majorant

$$C_R \sum_{r \geq 0} \frac{(r+1)^{r+4}}{r!} \{C_R(1+k)(|\zeta| + |\eta|)\}^r.$$

A radius smaller than $[2eC_R(1+k)]^{-1}$ proves (73). $\square$

Theorem 6.7 (Compact-time sextic asymptotic commutativity). *Let*

$$Y_{i,\sigma}(t) = e^{-itA_\sigma} \widetilde{X}_{i,\sigma} e^{itA_\sigma}.$$

For fixed k , a fixed bound on the number and total absolute time of sector propagators, and every $R < \infty$,

$$\sup_{\sigma, a \neq b, |s|, |t| \leq R, L} \frac{\mathbb{E} \operatorname{Tr}(e^{A_\sigma} L[Y_{a,\sigma}(s), Y_{b,\sigma}(t)][Y_{a,\sigma}(s), Y_{b,\sigma}(t)]^* L^*)}{\mathbb{E} \operatorname{Tr} e^{A_\sigma}} \longrightarrow 0. \quad (75)$$

The convergence is uniform in $\theta \in [0, 1]$ for every fixed atomic bath.

Proof. Suppose uniform convergence failed. Choose witnessing n_j , sectors, labels, history words, and real points in the compact rectangle, and pass to a subsequence on which the real points converge. Lemma 6.6 makes the associated entire functions a normal family on a neighborhood of every point in the rectangle. A finite chain of overlapping bidiscs joins the origin bidisc to a neighborhood of the limiting witnessing point; a diagonal Montel extraction gives a holomorphic limit on their connected union. Lemma 6.5 makes that limit identically zero on the origin bidisc. The several-complex-variables identity theorem therefore makes it zero throughout the connected union, contradicting the positive real value at the witnessing point. Equation (68) identifies the real restriction with the square in (75).

All constants above depend on the hybridization only through the fixed bounds Λ_B, Λ_Z . In the finite atomic interpolation bath those bounds are uniform in n , the sector, and $\theta \in [0, 1]$. This proves the final uniformity assertion. $\square$

6.4 Fixed-mark modular transfer and the slow triangular diagonal

The operator-algebraic part of the argument has two logically distinct limits. First one fixes the number of marked coordinates and takes $N \rightarrow \infty$; only then does one let the number of marked coordinates increase along a sufficiently slow diagonal. We state and prove the transfer theorem in that two-stage form.

For a marked set I of cardinality m , let $A_{N,I,\theta,\sigma}$ denote the deleted generator (51), with every sextet meeting I in at least two sites removed. Let $\Phi_{N,I,\theta,\sigma}$ be the corresponding annealed state. For fixed history complexity a and time bound T , define

$$\eta_{N,m}(a, T) := \sup_{\substack{\theta \in [0,1], \sigma, i \neq j \\ |s|, |t| \leq T, L \in \mathfrak{L}_{a,T}}} \Phi_{N,I,\theta,\sigma}(L[Y_{i,\sigma}(s), Y_{j,\sigma}(t)][Y_{i,\sigma}(s), Y_{j,\sigma}(t)]^* L^*)^{1/2}. \quad (76)$$

Theorem 6.7 gives $\eta_{N,m}(a, T) \rightarrow 0$ for every fixed m, a, T , uniformly in θ and in the labels.

Lemma 6.8 (Uniform field modulus for every ambient marked count). *Let*

$$X_{i,\sigma} = \varepsilon_{i,\sigma}(R_i + Z_i), \quad R_i = \gamma_N \sum_{T \in \binom{\Lambda}{5}} g_{i,T} \Psi_{\{i\} \cup T},$$

where the $g_{i,T}$ are independent standard Gaussians, $\sup_N \gamma_N^2 \binom{|\Lambda|}{5} < \infty$, and $\sup_i \|Z_i\| \leq \Lambda_Z$. For every compact real-time interval and every sector-propagator history H of fixed complexity and bounded total time, there is $C_H < \infty$, independent of N , $|I|$, θ , all sectors, and $i \in I$, such that

$$\Phi_\tau(HY_{i,\sigma}(r)^2 H^*) \leq 2\gamma_N^2 \binom{|\Lambda|}{5} + 2C_H \gamma_N^4 \binom{|\Lambda|}{5}^2 + 2\Lambda_Z^2 \leq C'_H. \quad (77)$$

Consequently, with $U_{i,\sigma}(t) = e^{itA_\sigma + e_i} e^{-itA_\sigma}$ and $\Phi_{i,\sigma} = \frac{1}{2}(\Phi_\sigma + \Phi_{\sigma+e_i})$,

$$\|U_{i,\sigma}(t) - I\|_{2, \Phi_{i,\sigma}} \leq C|t|, \quad |t| \leq 1, \quad (78)$$

with C independent of the ambient marked count.

Proof. Put $L = H e^{-irA_\sigma}$. Since L is unitary and $(R_i + Z_i)^2 \leq 2R_i^2 + 2Z_i^2$, it suffices to bound the random part. Expanding it gives

$$\gamma_N^2 \sum_{T,U} \mathbb{E} \operatorname{Tr}(e^{A_\tau} L g_{i,T} g_{i,U} \Psi_{\{i\} \cup T} \Psi_{\{i\} \cup U} L^*).$$

Twice applying Gaussian integration by parts yields a Kronecker term and a two-derivative term. For $T = U$ the Clifford unitary cancels. The two derivatives of the complete alternating heat/real-time trace word are bounded pointwise by $C_H \gamma_N^2 \operatorname{Tr} e^{A_\tau}$ by the trace-word derivative estimate [4, Lemma 4.1]; all disorder-dependent propagators are included in that word. Division by the common annealed partition function gives the first inequality in (77). Its right side is uniformly bounded by the covariance-mass hypothesis.

The cocycle evolution equation is

$$C_{i,\sigma}(t) - I = -2i \int_0^t Y_{i,\sigma}(r) C_{i,\sigma}(r) \, dr, \quad C_{i,\sigma}(t) = e^{-itA_\sigma} e^{itA_{\sigma+e_i}}.$$

Minkowski's inequality and cancellation of the unitary suffix in the right L^2 norm give a $C|t|$ bound in each of the two adjacent sector states. Since $U_{i,\sigma}(t) = C_{i,\sigma}(-t)^*$ and $(V - I)(V - I)^* = (V^* - I)(V^* - I)^*$ for a unitary V , this proves (78). $\square$

Proposition 6.9 (Fixed-mark modular transfer). *Fix $m < \infty$, $\beta < \infty$, a finite atomic bath, and an arbitrary sequence $N_j \rightarrow \infty$ with $\theta_j \in [0, 1]$. In the m -marked deleted sextic sign systems, pass to any subsequence on which the countable rational-time history moments converge and then to a Groh–Raynaud ultraproduct. The following assertions hold.*

- (F1) *All finite sign-sector ultraproduct states have one common support projection p . On pMp , the compressed finite-dimensional cocycles are the Connes cocycles and are strongly continuous.*
- (F2) *For a fixed base sector, the von Neumann algebras generated by distinct coordinate cocycles commute, are globally invariant under the base modular group, and obey the local sign identities*

$$\alpha_i(\mathcal{B}_i) = \mathcal{B}_i, \quad \alpha_i|_{\mathcal{B}_j} = \text{id} \quad (i \neq j).$$

- (F3) *The restriction of the faithful base state to every finite join $\bigvee_{i \in F} \mathcal{B}_i$ factors after central disintegration into product states across the coordinate algebras.*

Every assertion is uniform in the prescribed interpolation parameters θ_j .

Proof. The commuting trace-preserving involutions $\alpha_i = \text{Ad}(\mathbf{q}_i)$ satisfy exactly

$$\alpha_i(A_\sigma) = A_{\sigma+e_i}, \quad \alpha_i(X_{i,\sigma}) = -X_{i,\sigma}, \quad \alpha_i(X_{j,\sigma}) = X_{j,\sigma} \quad (i \neq j).$$

The free bath is fixed and the local hybridization at site i changes sign because $\mathbf{q}_i$ anticommutes with its attached odd bath Majorana. Thus the abstract sign-system relations [4, (4.41)–(4.46)] hold verbatim.

Lemma 4.12 of [4] is an algebraic Duhamel identity for any such sign system. For example,

$$[C_i(s), C_j(t)] = -4 \int_0^s \int_0^t C_i(s, r) C_j(t, u) [Y_i(r), Y_j(u)] C_j(u) C_i(r) \, du \, dr,$$

and the two-flip product defect has the analogous triangular double integral. Multiplying on the left by a fixed history, using the right-GNS orientation to cancel every unitary suffix, and applying Minkowski shows that all commutator, cocycle-product, and off-coordinate sign-action defects vanish by (76). This is exactly the implication proved in [4, Proposition 4.13]; no interaction label appears in that implication.

Lemma 6.8 verifies the sole model-dependent hypothesis of the relative-modular compactness step. For two faithful finite-dimensional sector states interchanged by a trace-preserving involution, [4, Lemma 4.15] identifies the symmetrized relative-modular spectral measure ν and proves from (78) the quantitative tail

$$\nu\{|x| \geq R\} \leq \frac{C^2}{6(1 - \sin 1)R^2}, \quad R \geq 1.$$

Thus the measures are tight. Proposition 4.16 of [4] then gives equal supports for each adjacent pair of ultraproduct states; connectivity of the finite m -cube gives one common support. The ultraproduct form of Raynaud’s L^1 cocycle theorem, [9, Proposition 2.2(i)], as used in [4, Theorem 4.17], identifies the compressed matrix cocycles with the Connes cocycles. This proves (F1).

Compress the vanishing defects to the common faithful corner. A zero right-GNS norm is now an operator equality. The Connes chain rule gives modular invariance of each coordinate algebra. These steps are precisely [4, Proposition 4.18] and prove (F2). For (F3), do not apply Lemma 4.19 to the possibly nonseparable ambient ultraproduct corner. Instead let $\mathcal{N}_F$ be the von Neumann algebra, in the cyclic GNS representation of the restricted faithful state, generated by the rational-time cocycles of a fixed finite coordinate set F . Its defining C^* -algebra is separable, so the cyclic Hilbert space is separable; $\mathcal{N}_F$ is therefore separably acting and σ -finite. The coordinate subalgebras inside $\mathcal{N}_F$ are countably generated, pairwise commuting, and globally modularly invariant. These are exactly the hypotheses of [4, Lemma 4.19]. The state-preserving expectations use Takesaki’s conditional-expectation theorem [10]; their measurable fiber reduction uses [11, Chapter IV, Section 8, especially Theorems 8.21

and 8.23]. Takesaki expectations on the factor fibers of its central disintegration successively factor all mixed moments. This proves (F3) for each fixed finite F on the rational-time history core; normality and strong continuity extend it to the real-time coordinate algebras used below. No compatibility of the central decompositions for different F is assumed or needed: the single projective directing law is constructed later by Lemma 5.2 of [4]. Every input estimate above is uniform in θ_j . $\square$

Theorem 6.10 (Triangular de Finetti and contour transfer). *Fix $\beta < \infty$ and one finite atomic bath. There is a deterministic integer sequence $k_N = k_N(\beta, \text{bath}) \uparrow \infty$ with $k_N^2/N \rightarrow 0$ such that the following holds. Given arbitrary $N_j \rightarrow \infty$ and $\theta_j \in [0, 1]$, one may pass to a subsequence for which the k_{N_j} -marked deleted sextic models have simultaneously:*

- (T1) *convergence of every word in a countable dense algebra generated by rational-time finite-sector cocycles on finitely many coordinates;*
- (T2) *commuting, exchangeable coordinate von Neumann algebras on one common faithful support, with the modular and sign-locality properties of Proposition 6.9;*
- (T3) *one probability measure Π on the compact metrizable state space of the separable one-coordinate rational-time algebra such that*

$$\Phi_{\text{cav}}|_{\mathcal{A}_{\text{hist}}} = \int \nu^{\otimes \mathbb{N}_+} \Pi(d\nu). \quad (79)$$

For Π -almost every ν , the product state annihilates the kernel of the concrete maximal- tensor-product quotient, and the same is true after every finite coordinate sign action;

- (T4) *the same Π -directed product law for every fixed family of moving labels outside the marked set, uniformly in their actual distinct labels and locally uniformly in compact real-time sets;*
- (T5) *complex-contour factorization for the precise incidence class used in the cavity expansion, locally uniformly on every compact subset of one orientation cell: the nondistinguished moving coordinates are pairwise distinct, each changes sign exactly twice, and there is no contraction between two nondistinguished coordinates. All explicit contractions and local insertions remain in one fixed distinguished subsystem. A moving coordinate with oriented separation w then has conditional factor $2\tilde{G}_\nu(w)$, where*

$$G_\nu(z) = \int_0^\infty \frac{\cosh(E(1/2 - z))}{2 \cosh(E/2)} \rho_\nu(dE), \quad 0 \leq \Re z \leq 1, \quad (80)$$

for a probability measure ρ_ν , and $\tilde{G}_\nu$ is the oriented antiperiodic continuation. Distinct moving coordinates factor conditionally under ν .

Every fixed contour word in the original one-site cavity model has the same limit as its representative on this triangular deleted sequence. All conclusions hold along prescribed parameter sequences θ_j , including sequences converging to an endpoint of $[0, 1]$.

Proof. The diagonal construction is the one step where the fixed-mark statements cannot be applied directly, so we give it in full. Enumerate two countable lists, arranging that the ℓ th item uses only the first ℓ coordinates: (i) normalized rational-time history moments; and (ii) the finite- N nonnegative defects consisting of the field-commutator squares in (76), the commutator, two-flip-product, and off-coordinate sign-action defects obtained from the Duhamel identities of Lemma 4.12 of [4], and the deletion defects in (53) and (54). Common support and modular identification are deliberately not put on this finite- N error list; they are obtained after the scalar diagonal from the uniform field modulus. For fixed r , evaluate the first r defects in the full r -marked deleted model and let $d_r(N)$ be their maximum after taking the supremum over $\theta \in [0, 1]$. Theorem 6.7, the Duhamel bounds in Proposition 4.13 of [4], and Proposition 6.1 give

$$d_r(N) \longrightarrow 0 \quad (N \rightarrow \infty), \quad r \text{ fixed}. \quad (81)$$

The constants may grow with r ; this causes no problem because r is fixed before the limit.

Choose increasing thresholds N_r so that for $N \geq N_r$, $d_r(N) \leq r^{-1}$ and $r^2/N \leq r^{-1}$. Define $k_N = r$ for $N_r \leq N < N_{r+1}$, altering finitely many initial values arbitrarily. Then $k_N \rightarrow \infty$, $k_N^2/N \rightarrow 0$, and every listed defect whose coordinates are eventually present vanishes. All normalized history moments are uniformly bounded by generalized Hölder. A scalar diagonal subsequence makes the entire countable family converge. For every fixed finite sector, Lemma 6.8 is uniform in the ambient marked count k_N ; hence the symmetrized relative-modular spectral measures for each adjacent sector pair are tight along the scalar diagonal. Proposition 4.16 and Theorem 4.17 of [4] then give one common support and identify every fixed-coordinate matrix cocycle with the corresponding Connes cocycle. The vanishing finite- N defects, compressed to that faithful corner, give the commuting, modular, and sign-locality conclusions exactly as in Proposition 4.18 of [4]. Fixed finite permutations act inside the marked set on a tail, so permutation invariance of the deleted model gives exchangeability. This proves (T1)–(T2).

Apply the product central disintegration of Proposition 6.9 to each fixed finite coordinate family. The projective lemma [4, Lemma 5.2] then produces one measure Π satisfying (79). Its proof averages the finite-fiber one-coordinate states and uses that the fraction of colliding indices in a degree- r statistic is at most $r(r-1)/(2k_N)$. Quotient descent follows from positivity: if $q_{\max}$ is the concrete quotient and $\{x_n\}$ is a countable norm-dense subset of $\ker q_{\max}$, then

$$0 = \Phi(q_{\max}(x_n^* x_n)) = \int \nu^{\otimes \mathbb{N}_+}(x_n^* x_n) \Pi(d\nu).$$

After one common null set, Cauchy–Schwarz makes every product state annihilate the entire kernel. Equivariance of the quotient gives the same conclusion after finite sign actions. This proves (T3).

Let F_N be any set of r additional moving labels, with r fixed. Delete additionally on $I_N \cup F_N$. Proposition 6.1 changes a fixed contour word by at most

$$C_W \frac{k_N r + r^2}{N} = o(1)$$

uniformly in the labels. The enlarged model is exchangeable. Symmetrizing a fixed degree- s statistic over $I_N \setminus F_N$ instead of I_N costs $O_{r,s}(k_N^{-1})$; the fraction of tuples with repeated indices is also $O_s(k_N^{-1})$. Therefore the moving coordinates have the same Π -directed product law as the marked coordinates. A violating sequence of labels would admit the same diagonal extraction and contradict this conclusion, proving uniformity. The modulus (78) and telescoping give local uniformity in real time. This proves (T4).

It remains to justify (T5), including the positive Lehmann form. Fix the stated incidence class: every nondistinguished moving coordinate is distinct, occurs through exactly two sign changes, and there is no nondistinguished contraction. This restriction is essential and is precisely the class in Corollary 5.12 of [4]; an arbitrary contour word need not factor coordinate by coordinate. A multisegment heat/real-time word in this class is a bounded holomorphic function on the closed simplex tube by generalized Hölder. On each vertex tube, cyclically moving the unique heat factor first reduces the word to rational-time relative cocycles; the real-time modulus gives locally uniform boundary convergence. The strip and simplex-tube uniqueness lemmas [4, Lemmas 5.5–5.6] therefore extend the same limit to the heat interior. The normal sign actions fix the diagonal de Finetti parameter, and the support, relative modular operators, and Connes cocycles disintegrate measurably over Π by [4, Lemmas 5.7–5.9]. On the nondistinguished coordinate set, the fiber state and relative modular operator are tensor products, while every contraction and local insertion is retained in the distinguished word. Thus each distinct nondistinguished coordinate which changes twice contributes one scalar coefficient

$$K_\nu(z) = \langle \xi_\nu, \Delta_{\nu \circ \alpha|_\nu}^z \xi_\nu \rangle.$$

The involution interchanging the two one-coordinate states gives $K_\nu(z) = K_\nu(1-z)$. If μ_ν is the spectral measure of $\log \Delta_{\nu \circ \alpha|_\nu}$, then $(1+e^x)\mu_\nu(dx)/2$ is a symmetric probability measure. Pushing it forward

under $x \mapsto |x|$ gives the probability ρ_ν and the representation $K_\nu(z) = 2G_\nu(z)$ in (80). This is the argument of [4, Proposition 5.10] and is independent of the interaction arity.

The finite generator is even. Hence the graded KMS identity [4, Lemma 5.11] gives one minus sign whenever an odd coordinate implementer crosses the thermal cut. Consequently the oriented lift is antiperiodic. Applying the tube identity theorem on each orientation cell and then the boundary-layer estimate of [4, Corollary 5.12 and (5.42)] proves locally uniform complex-contour factorization. For a sextic cavity chord there are exactly five distinct moving bulk coordinates, hence five factors $2\tilde{G}_\nu$; this is the only place where the number six enters this final transfer step.

Finally, the original and triangular deleted models differ by $O_W(k_N^2/N) = o(1)$ on every fixed contour. Enlarging by finitely many moving coordinates costs the already controlled $O_W((k_N r + r^2)/N)$. Thus every conclusion returns to the original cavity model. All estimates used in constructing $d_r(N)$ were uniform in θ , so a prescribed parameter sequence, including endpoint limits, is covered. $\square$

Technical remark 6.11 (Dependencies of the modular transfer argument).

Source result	Abstract input	Sextic discharge
Lemma 4.12, Proposition 4.13	Sign-sector ODEs and compact-time field-commutator decay	Sign identities above and Theorem 6.7.
Lemmas 4.14–4.15	Random marked field with bounded covariance mass; bounded local part; linear cocycle L^2 modulus	Lemma 6.8.
Proposition 4.16, Theorem 4.17	Tight symmetrized relative-modular measures and faithful finite states	Quantitative R^{-2} tail and positivity of e^{A_σ} .
Proposition 4.18, Lemma 4.19	Common faithful support; vanishing cocycle defects; modular invariance	Proposition 6.9.
Lemma 5.1	Additional deletion on moving labels	Replaced by the sextic bound (54).
Lemmas 5.2–5.3, Proposition 5.4	Finite-family product disintegrations, exchangeability, and a slow diagonal	The explicit threshold construction following (81).
Lemmas 5.5–5.9, Proposition 5.10	Bounded tube holomorphy, fiber localization, and tensorization	The proof of (T5), using the common support and one measure Π .
Lemma 5.11, Corollary 5.12	Even generator, odd coordinate implementers, two transitions per coordinate	Unchanged; a sextic chord supplies five coordinate factors.

All arity-dependent inputs occur before this table and are verified with the corresponding fixed- q label counts.

7 The sextic single-site cavity expansion

Fix $\beta < \infty$, $\theta \in [0, 1]$, and one atomic source Σ_M . Distinguish physical site 0 and delete every interaction incident to it. The removed field in the dimensionless interpolation exponent is

$$X_0 = \gamma_N \sum_{T \in \binom{[N-1]}{5}} g_T \Psi_{\{0\} \cup T}, \quad \gamma_N^2 = \theta \beta^2 \frac{15}{8N^5}. \quad (82)$$

It is independent of the cavity exponent B . Put

$$b_N := \binom{N-1}{5}, \quad v_N := \gamma_N^2 b_N \leq \frac{\beta^2}{64}. \quad (83)$$

The local atomic bath at site 0 remains in B .

7.1 Summability and collision removal

Lemma 7.1 (Uniform factorial tail). *Consider a fixed mixed heat/real-time contour word whose heat lengths are nonnegative and sum to one, whose total absolute real-time length is at most T , and whose remaining insertions are contractions independent of the g_T 's. After expanding all contour segments in X_0 and averaging the g_T 's, the sum of the absolute values of all order- $2p$ terms, divided by the cavity partition function, is at most*

$$\frac{(2p-1)!!}{(2p)!} \{(1+T)^2 v_N\}^p = \frac{\{(1+T)^2 v_N\}^p}{2^p p!} \leq \frac{\{\beta^2(1+T)^2/128\}^p}{p!}. \quad (84)$$

The estimate remains valid with an additional cavity-measurable contraction.

Proof. Wick's rule gives $(2p-1)!!$ pairings. Each pair identifies two five-set labels, leaving at most b_N^p assignments and the factor γ_N^{2p} . The sum of the ordered-simplex volumes obtained by distributing $2p$ insertions among all contour segments is bounded by $(1+T)^{2p}/(2p)!$. Generalized Schatten Hölder bounds every trace integrand by the cavity partition function because the heat exponents sum to one, real-time factors are unitary, and all other insertions are contractions. Multiplication gives (84). The last inequality is (83). $\square$

Lemma 7.2 (Collision union bound). *Fix a Wick pairing with p chords and s distinct test coordinates. Assign one five-set to each chord. The fraction of assignments in which two five-sets intersect or a five-set meets a test coordinate is at most*

$$\frac{25p(p-1) + 10ps}{2(N-1)}. \quad (85)$$

After summing every Duhamel order, all collision assignments contribute $O_{\beta,T,s}(N^{-1})$ to the normalized contour word.

Proof. Expose the five-sets sequentially. After r sets, at most $5r + s$ forbidden sites have been exposed. A uniformly chosen five-set meets them with probability at most $5(5r + s)/(N-1)$ by the union bound over its five elements. Summing over $0 \leq r < p$ gives (85). Multiplying this polynomial in p by the summable majorant (84) and summing over p proves the second assertion. $\square$

7.2 The fermionic sign and the factor 64

Let $P_0 = \mathbf{q}_0$ and, for a five-set T , put

$$Q_T := \Psi_{\{0\} \cup T}, \quad R_T := P_0 Q_T. \quad (86)$$

Lemma 7.3 (Sextic cavity sign). *For pairwise disjoint five-sets T, U ,*

$$Q_T^2 = I, \quad Q_T Q_U = -Q_U Q_T, \quad R_T^2 = -I, \quad R_T R_U = -R_U R_T. \quad (87)$$

For a pairing $\mathbf{p}$ of $2p$ ordered cavity vertices, the residual distinguished-site word obtained after stripping the pairwise-disjoint bulk coordinates has CAR sign

$$(-1)^{\text{cr}(\mathbf{p})}, \quad (88)$$

where $\text{cr}(\mathfrak{p})$ is the number of chord crossings. This is exactly the Pfaffian sign of one quadratic fermionic bath. No additional sextic phase remains.

Proof. Two supports $\{0\} \cup T$ and $\{0\} \cup U$ intersect in one site, so (2) gives the first anticommutation relation. Since $P_0 Q_T = -Q_T P_0$, one has $R_T^2 = P_0 Q_T P_0 Q_T = -I$. Directly from (1), if $T = \{t_1 < \dots < t_5\}$,

$$R_T = -i \mathbf{q}_{t_1} \cdots \mathbf{q}_{t_5}. \quad (89)$$

Disjoint fivefold products are odd and anticommute, proving the remaining relation in (87).

First suppose there are no additional odd local insertions. Bring the two endpoints of every chord together in the ordered product of the Q_T 's. A transposition is needed precisely when two chords cross, and every such transposition exchanges anticommuting monomials. Each adjacent pair then gives $Q_T^2 = I$, yielding (88).

For the sign comparison with a quadratic bath in the presence of parity-homogeneous local insertions, write $Q_T = P_0 R_T$. Let E be, modulo two, the number of passages of the $2p$ odd R -blocks across later odd distinguished insertions. Moving the R -blocks to the bulk side contributes $(-1)^{\binom{2p}{2} + E}$. Their paired product contributes $(-1)^{p + \text{cr}(\mathfrak{p})}$: one factor -1 comes from each R_T^2 , and one from every crossing. Since $\binom{2p}{2} \equiv p \pmod{2}$, the total is $(-1)^{E + \text{cr}(\mathfrak{p})}$.

A quadratic hybridization vertex is $-i\chi_0\eta$, with η an odd bath Majorana. Moving all η 's past the same later local insertions contributes $(-1)^E$; its quasifree Wick rule contributes the Pfaffian sign $(-1)^{\text{cr}(\mathfrak{p})}$. The signs agree. The phase $-i$ in (89) is already responsible for $R_T^2 = -I$ and produces no further factor. $\square$

Lemma 7.4 (Quadratic-source matching). *Condition on a directing kernel G_λ . At one disjoint Wick chord, the sextic cavity expansion contributes*

$$v_N \{2G_\lambda\}^5 = 32 v_N G_\lambda^5. \quad (90)$$

A quadratic Majorana bath with added source sG_λ^5 contributes $(s/2)G_\lambda^5$ per chord in the normalized P_0 word. Hence the matching source strength is

$$s_N = 64 v_N, \quad 64 \gamma_N^2 \binom{N-1}{5} = \theta \beta^2 \alpha_{N,6}. \quad (91)$$

In particular $s_N \rightarrow \theta \beta^2$.

Proof. Theorem 6.10(T5) gives one factor $2G_\lambda$ for each of the five distinct bulk coordinates belonging to a chord. The label sum and Gaussian covariance give v_N , proving (90).

For a quadratic bath, each pair of raw physical fields contributes $\chi_0^2 = P_0^2/2$, so a source sG_λ^5 contributes $(s/2)G_\lambda^5$. Equating this with (90) gives $s_N = 64 v_N$. Finally,

$$\begin{aligned} 64 \gamma_N^2 \binom{N-1}{5} &= 64 \theta \beta^2 \frac{15}{8N^5} \frac{(N-1)(N-2)(N-3)(N-4)(N-5)}{120} \\ &= \theta \beta^2 \prod_{j=1}^5 \left(1 - \frac{j}{N}\right), \end{aligned}$$

which is (91). $\square$

7.3 Cavity restoration in the directing-state disintegration

Theorem 7.5 (Sextic cavity identification). *Given arbitrary $N_j \rightarrow \infty$ and $\theta_j \rightarrow \theta \in [0, 1]$, pass to the subsequence and directing law π_{cav} supplied by Theorem 6.10. Restoring X_0 has the following conditional limit. For π_{cav} -almost every directing state λ , the distinguished local system becomes a quadratic fermionic bath state with source*

$$\Sigma_{\theta, \lambda} = (1 - \theta)\Sigma_M + \theta\beta^2 G_\lambda^5. \quad (92)$$

Its partition-function ratio is $\mathcal{D}(\Sigma_{\theta, \lambda})$, and its two-state kernel is $D(\Sigma_{\theta, \lambda})$. The restored directing law is

$$\tilde{\pi}(\mathrm{d}\lambda) = \frac{\mathcal{D}(\Sigma_{\theta, \lambda})\pi_{\text{cav}}(\mathrm{d}\lambda)}{\int \mathcal{D}(\Sigma_{\theta, \lambda})\pi_{\text{cav}}(\mathrm{d}\lambda)}. \quad (93)$$

It is equivalent to π_{cav} and obeys

$$G_\lambda = D((1 - \theta)\Sigma_M + \theta\beta^2 G_\lambda^5) \quad \text{for } \tilde{\pi}\text{-almost every } \lambda. \quad (94)$$

Consequently G_λ is almost surely the deterministic unique solution $G_{M, \theta}^\circ$ of (94).

Proof. Fix a contour cutoff $2P$. Wick-expand the restored field. Lemma 7.2 removes intersecting five-sets and five-sets meeting the fixed test support. For every remaining assignment, the $5p$ bulk coordinates and finitely many test coordinates are distinct. Enlarging the triangular deletion from the marked set I to their union changes the word by

$$O_{P, s, \beta, M, T} \left(\frac{k^2 + k(5P + s) + (5P + s)^2 + P^2 + Ps}{N} \right), \quad (95)$$

by Proposition 6.1 and Lemma 7.2. This tends to zero because $k^2/N \rightarrow 0$.

At fixed order, Theorem 6.10(T5) factors the $5p$ coordinates conditionally on λ . Lemma 7.3 identifies the Pfaffian sign, and Lemma 7.4 identifies the source coefficient. Thus every order up to $2P$ converges to the corresponding coefficient of the local quadratic state with source (92). Lemma 7.1 bounds the original tail uniformly in N, θ ; the Pfaffian series has the same factorial majorant because $|G_\lambda| \leq 1/2$ and the total source mass is $\beta^2/16$. Letting $P \rightarrow \infty$ proves equality of the full contour series.

The determinant and local contour state are continuous under positive-source approximation by Lemma 4.2. The source in (92) has mass

$$(1 - \theta)\frac{\beta^2}{16} + \theta\beta^2\frac{1}{16} = \frac{\beta^2}{16}.$$

Therefore

$$1 \leq \mathcal{D}(\Sigma_{\theta, \lambda}) \leq e^{\beta^2/128}, \quad (96)$$

which proves positivity, integrability, and equivalence of the two laws in (93).

It remains to identify the distinguished kernel with the bulk kernel. Choose one unused bulk coordinate i and finitely many further test coordinates with rational thermal arc lengths. The transposition $0 \leftrightarrow i$ leaves the restored finite- N annealed model invariant. Expand both observables around the site-0 cavity, use the preceding fixed-order comparison, and then remove the factorial tails. The resulting identity is

$$\int \mathcal{D}(\Sigma_{\theta, \lambda}) \{D(\Sigma_{\theta, \lambda})(u) - G_\lambda(u)\} \prod_{r=1}^s G_\lambda(v_r) \pi_{\text{cav}}(\mathrm{d}\lambda) = 0 \quad (97)$$

for rational $u, v_1, \dots, v_s \in (0, 1)$; repeated values are represented by distinct coordinates. Let $Q_0 = \mathbb{Q} \cap (0, 1)$ and put

$$\kappa(\lambda) = (G_\lambda(v))_{v \in Q_0}, \quad X = \overline{\kappa(\text{supp } \pi_{\text{cav}})} \subset [0, 1/2]^{Q_0}.$$

The space X is compact metrizable. Equip the normalized positive-Lehmann kernels with the Borel structure generated by rational evaluations; this is their compact-open Borel structure. The Lehmann transform is Borel and injective: injectivity follows by analytic continuation to the central line and uniqueness of the cosine transform. The Lusin–Souslin theorem [13, Theorem 15.1] therefore makes its inverse Borel on its image. The fifth-power convolution, addition of the fixed source, the Dyson map, and $\mathcal{D}$ are Borel on the bounded-mass class by Lemmas 4.1 and 4.2. Consequently both $\mathcal{D}(\Sigma_{\theta,\lambda})$ and $D(\Sigma_{\theta,\lambda})(u)$ are Borel functions of $\kappa(\lambda)$.

Let Y be the Borel image in $[0, 1/2]^{\mathcal{Q}_0}$ of the normalized positive-Lehmann kernel space under rational evaluation. The preceding Lusin–Souslin argument shows that Y is Borel and that the displayed weight is a Borel function of the evaluation vector. For rational u , define the finite signed measure μ_u on the compact space X as the pushforward

$$\mu_u(B) := \int_{\{\kappa(\lambda) \in B\}} \mathcal{D}(\Sigma_{\theta,\lambda}) \{D(\Sigma_{\theta,\lambda})(u) - G_\lambda(u)\} \pi_{\text{cav}}(d\lambda), \quad B \in \mathcal{B}(X).$$

It is supported on $X \cap Y$. No assertion is made that an arbitrary closure point of X is itself a Lehmann kernel; the pushforward definition only uses actual evaluation vectors in Y . Equation (97) says that μ_u annihilates every polynomial in finitely many coordinate evaluations. Constants are included by taking $s = 0$, and powers are represented by distinct test coordinates with repeated arc lengths. These polynomials form a unital self-adjoint algebra separating points of X . Stone–Weierstrass gives $\mu_u = 0$. Because (96) is strictly positive, $D(\Sigma_{\theta,\lambda})(u) = G_\lambda(u)$ almost surely. Intersect over rational u and use closed-strip continuity to obtain (94).

Proposition 4.3, with source measure $(1 - \theta)\nu_M$ and nonlinear coefficient $\theta\beta^2$, gives a unique normalized positive-Lehmann solution. Hence the conditional kernel is deterministic. $\square$

8 Vanishing of the sextic interpolation remainder and the finite-temperature pressure

For a fixed atomic bath, set

$$h_M^\circ(u) := \left(\frac{\Sigma_M(u)}{\beta^2} \right)^{1/5}. \quad (98)$$

The limiting polynomial is the Bregman divergence of $x \mapsto x^6$:

$$\mathcal{B}_6(x, r) := x^6 - 6r^5x + 5r^6 = (x - r)^2 (x^4 + 2rx^3 + 3r^2x^2 + 4r^3x + 5r^4) \geq 0. \quad (99)$$

If $0 \leq x, r \leq 1/2$, then

$$0 \leq \mathcal{B}_6(x, r) \leq \frac{15}{16}(x - r)^2. \quad (100)$$

Proposition 8.1 (Fixed-bath subsequential remainder limit). *Fix M and let $N_j \rightarrow \infty$ and $\theta_j \in [0, 1]$ be arbitrary. There is a subsequence, still denoted j , and $\theta_j \rightarrow \theta_\infty$ such that*

$$\lim_{j \rightarrow \infty} \int_0^1 \mathcal{R}_{M, N_j}(u, \theta_j) du = \int_0^1 \mathcal{B}_6(G_{M, \theta_\infty}^\circ(u), h_M^\circ(u)) du, \quad (101)$$

where $G_{M, \theta}^\circ$ is the unique normalized positive-Lehmann solution of

$$G = D((1 - \theta)\Sigma_M + \theta\beta^2 G^5). \quad (102)$$

The limit is nonnegative.

Proof. Pass to a subsequence on which $\theta_j \rightarrow \theta_\infty$ and use the diagonal sequence in Theorem 6.10. Let I_j be the marked set, with $k_j \rightarrow \infty$ and $k_j^2/N_j \rightarrow 0$. For $r \in \{1, 6\}$ define the without-replacement coordinate averages

$$U_{j,r}(u) := \frac{1}{(k_j)_r} \sum_{i_1, \dots, i_r \in I_j}^{\text{distinct}} f_{\{i_1, \dots, i_r\}, M, \theta_j}(u).$$

Exchangeability and (40) give

$$U_{j,1} = 2G_{M, \theta_j}, \quad U_{j,6} = 64K_{6, M, \theta_j}. \quad (103)$$

On every compact subinterval $[\delta, 1 - \delta]$, Theorem 6.10(T3)–(T5), with the parameter-uniform cavity identification of Theorem 7.5, yields locally uniform convergence

$$U_{j,r}(u) \longrightarrow \int \{2G_\lambda(u)\}^r \tilde{\pi}_{M, \theta_\infty}(d\lambda), \quad r = 1, 6. \quad (104)$$

Here $\tilde{\pi}$ is the positive determinant reweighting in (93). Theorem 7.5 and Proposition 4.3 imply $G_\lambda = G_{M, \theta_\infty}^\circ$ almost surely.

For completeness, the passage from independent marked-coordinate sampling, which occurs in the projective de Finetti moments, to the six-distinct-site overlap in (103) has no hidden diagonal term. Lemma 6.2 gives an error at most $15/k_j$. Enlarging the deletion set by the at most six sampled coordinates contributes $O_{\beta, M}(k_j^2/N_j + k_j/N_j + N_j^{-1})$ by Proposition 6.1. Repeated-coordinate terms in the sixth power have the same $O(k_j^{-1})$ bound. Hence all three forms of the sixth moment have the common limit in (104).

Since $\alpha_{N_j, 6} \rightarrow 1$, the deterministic functions

$$h_{M, N_j}(u) = \left(\frac{\Sigma_M(u)}{\beta^2 \alpha_{N_j, 6}} \right)^{1/5}$$

converge to h_M° in every finite $L^p(0, 1)$. Equations (103)–(104) therefore send the polynomial in (44), on $[\delta, 1 - \delta]$, to

$$(G_{M, \theta_\infty}^\circ)^6 - 6(h_M^\circ)^5 G_{M, \theta_\infty}^\circ + 5(h_M^\circ)^6 = \mathcal{B}_6(G_{M, \theta_\infty}^\circ, h_M^\circ).$$

It remains only to remove the two thermal boundary intervals. For all sufficiently large j , $\alpha_{N_j, 6} \geq 1/2$. From (35), $0 \leq h_{M, N_j}^5 \leq 1/16$ and $0 \leq h_{M, N_j} \leq 1$; moreover $0 \leq G \leq 1/2$ and $0 \leq K_6 \leq 1/64$. Thus $|\mathcal{R}_{M, N_j}| \leq C$ with a numerical constant independent of u, j, θ_j . The limiting integrand is also uniformly bounded. Its two boundary pieces have total length 2δ . Letting first $j \rightarrow \infty$ and then $\delta \downarrow 0$ proves (101). Nonnegativity follows from (99). $\square$

Proposition 8.2 (Uniform removal of the finite bath). *Let $G_{\beta, 6}$ be the solution of (30). Then*

$$\sup_{0 \leq \theta \leq 1} \|G_{M, \theta}^\circ - G_{\beta, 6}\|_{L^p(0, 1)} \longrightarrow 0 \quad (1 \leq p < \infty). \quad (105)$$

Also

$$\|h_M^\circ - G_{\beta, 6}\|_{L^5}^5 \leq \beta^{-2} \|\Sigma_M - \beta^2 G_{\beta, 6}^5\|_{L^1} \longrightarrow 0. \quad (106)$$

Consequently,

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{0 \leq \theta \leq 1} \left| \int_0^1 \mathcal{R}_{M, N}(u, \theta) du \right| = 0. \quad (107)$$

Proof. For (105), argue by contradiction. Given arbitrary $M_j \rightarrow \infty$ and $\theta_j \in [0, 1]$, pass to a subsequence with $\theta_j \rightarrow \theta_\infty$. The source measures of $(1 - \theta_j)\Sigma_{M_j}$ converge weakly to that of $(1 - \theta_\infty)\beta^2 G_{\beta,6}^5$, and the nonlinear coefficients $\theta_j \beta^2$ converge to $\theta_\infty \beta^2$. Parameter continuity in Proposition 4.3 shows that G_{M_j, θ_j}° converges in every finite L^p to the unique solution of

$$G = D((1 - \theta_\infty)\beta^2 G_{\beta,6}^5 + \theta_\infty \beta^2 G^5).$$

The kernel $G_{\beta,6}$ solves this equation, and uniqueness identifies the limit. This proves uniformity.

For nonnegative a, b and $0 < r \leq 1$, $|a^r - b^r| \leq |a - b|^r$. Apply this with $r = 1/5$, $a = \Sigma_M/\beta^2$, and $b = G_{\beta,6}^5$ to obtain (106).

Equations (105) and (106), interpolation between L^p norms, and the common bound by $1/2$ imply that

$$\varepsilon_M := \sup_{0 \leq \theta \leq 1} \int_0^1 \mathcal{B}_6(G_{M,\theta}^\circ, h_M^\circ) du \longrightarrow 0; \quad (108)$$

one may use the explicit estimate (100).

Fix M for the moment and write $Q_{M,N}(\theta) = \int_0^1 \mathcal{R}_{M,N}(u, \theta) du$. We claim

$$\limsup_{N \rightarrow \infty} \sup_{0 \leq \theta \leq 1} |Q_{M,N}(\theta)| \leq \varepsilon_M. \quad (109)$$

If this failed, there would be $\delta > 0$, $N_j \rightarrow \infty$, and $\theta_j \in [0, 1]$ with $|Q_{M,N_j}(\theta_j)| > \varepsilon_M + \delta$. Proposition 8.1 would supply a further subsequence on which $\theta_j \rightarrow \theta_\infty$ and

$$Q_{M,N_j}(\theta_j) \longrightarrow \int_0^1 \mathcal{B}_6(G_{M,\theta_\infty}^\circ, h_M^\circ) du,$$

whose absolute value is at most ε_M , a contradiction. Thus (109) holds for every fixed M . Letting $M \rightarrow \infty$ in (109) and using (108) proves (107). $\square$

Theorem 8.3 (Finite-temperature sextic pressure). *For every fixed $\beta \geq 0$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \operatorname{tr} e^{\beta \mathcal{H}_{N,6}} = p_{\text{SD},6}(\beta) = \log \mathcal{D}(\beta^2 G_{\beta,6}^5) - \frac{5\beta^2}{12} \int_0^1 G_{\beta,6}(u)^6 du. \quad (110)$$

Proof. At $\beta = 0$ both sides vanish. Fix $\beta > 0$. Corollary 5.2 and Proposition 8.2 reduce the limit to the deterministic finite-bath expression

$$A_M := \log \mathcal{D}(\Sigma_M) - \frac{5}{12} \beta^{-2/5} \int_0^1 \Sigma_M(u)^{6/5} du, \quad (111)$$

because

$$\mathfrak{a}_{N,6} h_M^6 = \mathfrak{a}_{N,6}^{-1/5} \Sigma_M^{6/5} \longrightarrow \beta^{-2/5} \Sigma_M^{6/5}.$$

The convergences in (34) give

$$\log \mathcal{D}(\Sigma_M) \rightarrow \log \mathcal{D}(\beta^2 G_{\beta,6}^5)$$

and

$$\beta^{-2/5} \Sigma_M^{6/5} \rightarrow \beta^{-2/5} (\beta^2 G_{\beta,6}^5)^{6/5} = \beta^2 G_{\beta,6}^6$$

in L^1 . Therefore $A_M \rightarrow p_{\text{SD},6}(\beta)$, proving (110). $\square$

9 Zero-temperature sextic Schwinger–Dyson equation

Fix $\beta > 0$ and abbreviate $G_\beta = G_{\beta,6}$ and $\mu_\beta = \mu_{\beta,6}$. Let ρ_β be the pushforward of μ_β under $x \mapsto x/\beta$. By (31),

$$\rho_\beta([0, \infty)) = 1, \quad \int_0^\infty E^2 \rho_\beta(dE) = \frac{1}{16}. \quad (112)$$

For $0 \leq t \leq \beta/2$, put

$$K_{\beta,E}(t) := k_{\beta E}(t/\beta) = \frac{e^{-Et} + e^{-E(\beta-t)}}{2(1 + e^{-\beta E})}, \quad g_\beta(t) := G_\beta(t/\beta) = \int K_{\beta,E}(t) \rho_\beta(dE). \quad (113)$$

The fifth-power closure supplies a probability measure λ_β such that

$$16G_\beta(u)^5 = \int k_y(u) \lambda_\beta(dy). \quad (114)$$

Let ℓ_β be its pushforward under $y \mapsto y/\beta$. Define, for $v > 0$,

$$R_\beta(v) := \int \frac{v}{v^2 + E^2} \rho_\beta(dE), \quad R_{\ell,\beta}(v) := \int \frac{v}{v^2 + E^2} \ell_\beta(dE), \quad (115)$$

and the dimensionful fermionic grid

$$\mathcal{V}_\beta := \left\{ v_{\beta,n} = \frac{(2n+1)\pi}{\beta} : n \geq 0 \right\}. \quad (116)$$

Lemma 9.1 (Dimensionful finite-temperature Dyson identities). *For $v \in \mathcal{V}_\beta$, let*

$$S_\beta(v) := 2 \int_0^{\beta/2} \sin(vt) g_\beta(t)^5 dt. \quad (117)$$

Then

$$S_\beta(v) = \frac{1}{16} R_{\ell,\beta}(v), \quad R_\beta(v) = \frac{1}{v + S_\beta(v)}. \quad (118)$$

Proof. For $\omega_n = (2n+1)\pi$, the unit-circle Fourier identity for a Lehmann kernel is

$$\widehat{G}_\beta(n) = i\omega_n \int \frac{\mu_\beta(dx)}{\omega_n^2 + x^2}.$$

After $x = \beta E$ this is $iR_\beta(v_{\beta,n})/\beta$. The self-energy coefficient satisfies

$$\frac{s_{\beta,n}}{\beta} = -i \int_0^\beta e^{iv_{\beta,n}t} g_\beta(t)^5 dt = 2 \int_0^{\beta/2} \sin(v_{\beta,n}t) g_\beta(t)^5 dt,$$

where symmetry about $\beta/2$ and $e^{iv_{\beta,n}\beta} = -1$ were used. Division of the reciprocal Fourier Dyson identity by β gives the second equation in (118). Applying the same Fourier calculation to (114) gives $R_{\ell,\beta} = 16S_\beta$. $\square$

Theorem 9.2 (Uniform sextic tail). *There is a numerical constant $C < \infty$, independent of β , such that*

$$g_\beta(t)^6 \leq Ct^{-2}, \quad g_\beta(t) \leq Ct^{-1/3}, \quad 1 \leq t \leq \beta/2. \quad (119)$$

Consequently, for $T \geq 1$,

$$\sup_{\beta \geq 2T} \int_T^{\beta/2} g_\beta(t)^5 dt \leq CT^{-2/3}, \quad (120)$$

$$\sup_{\beta \geq 2T} \int_T^{\beta/2} g_\beta(t)^6 dt \leq CT^{-1}. \quad (121)$$

Proof. For $0 \leq t \leq \beta/2$ the image term in (113) is at most the direct term, hence

$$0 \leq K_{\beta,E}(t) \leq e^{-Et} \leq \frac{1}{1 + E^2 t^2}. \quad (122)$$

Fix $1 \leq t \leq \beta/2$. The spacing of the numbers vt , $v \in \mathcal{V}_\beta$, is at most π , so one can choose v with $1 \leq a := vt \leq 1 + \pi$. Uniformly for such a and all $x \geq 0$,

$$\frac{1}{1 + x^2} \leq C \frac{a}{a^2 + x^2}.$$

Apply this first to ρ_β and then to ℓ_β . Equations (113), (114), and (118) give

$$R_\beta(v) \geq ctg_\beta(t), \quad S_\beta(v) \geq ctg_\beta(t)^5. \quad (123)$$

The reciprocal equation now implies

$$ctg_\beta(t) \leq R_\beta(v) = \frac{1}{v + S_\beta(v)} \leq \frac{C}{tg_\beta(t)^5},$$

so $g_\beta(t)^6 \leq Ct^{-2}$. The remaining assertions follow by taking a sixth root and integrating $t^{-5/3}$ and t^{-2} . $\square$

Theorem 9.3 (Existence and normalization of the zero-temperature solution). *Every sequence $\beta_j \rightarrow \infty$ has a subsequence and a probability measure ρ on $[0, \infty)$ such that*

$$\rho_{\beta_j} \Rightarrow \rho, \quad g_{\beta_j}(t) \longrightarrow g(t) := \frac{1}{2} \int_0^\infty e^{-Et} \rho(dE) \quad (124)$$

locally uniformly for $t \geq 0$. The limit satisfies

$$\int E^2 \rho(dE) = \frac{1}{16}, \quad g^5 \in L^1(0, \infty), \quad (125)$$

and, for every $v > 0$,

$$R(v) := \int \frac{v}{v^2 + E^2} \rho(dE) = \frac{1}{v + S(v)}, \quad S(v) := 2 \int_0^\infty \sin(vt) g(t)^5 dt. \quad (126)$$

Proof. The common second moment in (112) makes the probability measures ρ_β tight. Prokhorov's theorem gives a weakly convergent subsequence. For fixed $T < \infty$,

$$K_{\beta,E}(t) - \frac{1}{2} e^{-Et} = \frac{e^{-\beta E} \sinh(Et)}{1 + e^{-\beta E}}, \quad 0 \leq t \leq T. \quad (127)$$

Writing $a = t/\beta$ and $u = \beta E$, the numerator is bounded by $e^{-u} \sinh(au) \leq aue^{-(1-a)u} \leq C_T/\beta$ once $\beta \geq 2T$. Thus the error tends to zero uniformly in E and $t \in [0, T]$. Weak convergence proves pointwise convergence in (124). Moreover,

$$|\partial_t K_{\beta,E}(t)| \leq E/2, \quad \int E \rho_\beta(dE) \leq 1/4,$$

so the functions are equi-Lipschitz. The convergence is locally uniform.

The tail estimate (120), together with local uniform convergence, gives $g^5 \in L^1$ and uniform control of its tails. Given $v > 0$, choose $v_j \in \mathcal{V}_{\beta_j}$ with $v_j \rightarrow v$. Weak convergence gives $R_{\beta_j}(v_j) \rightarrow R(v)$; local convergence plus (120) gives $S_{\beta_j}(v_j) \rightarrow S(v)$. Lemma 9.1 yields (126).

Weak convergence initially gives only $\int E^2 \rho \leq 1/16$. This still implies a finite first moment, hence g and $h := g^5$ are absolutely continuous. The function h is decreasing, integrable, and $h(0) = 2^{-5} = 1/32$; therefore $\int_0^\infty |h'(t)| dt = h(0)$. Integration by parts and the Riemann–Lebesgue lemma give

$$vS(v) = 2h(0) + 2 \int_0^\infty \cos(vt)h'(t) dt \longrightarrow \frac{1}{16}. \quad (128)$$

Consequently

$$R(v) = \frac{1}{v + S(v)} = \frac{1}{v} - \frac{1}{16v^3} + o(v^{-3}).$$

On the other hand, monotone convergence gives

$$v^3 \left(\frac{1}{v} - R(v) \right) = \int \frac{v^2 E^2}{v^2 + E^2} \rho(dE) \longrightarrow \int E^2 \rho(dE).$$

Comparison proves the equality in (125). $\square$

Theorem 9.4 (Uniqueness of the normalized sextic zero-temperature solution). *There is at most one probability measure satisfying (125)–(126) with the positive Laplace representation in (124).*

Proof. Let (ρ_j, g_j, R_j, S_j) , $j = 1, 2$, be two solutions, and put

$$f = g_1 - g_2, \quad h = g_1^5 - g_2^5, \quad F = R_1 - R_2, \quad H = S_1 - S_2.$$

The functions F and H are twice the sine transforms of f and h , respectively; the first transform is understood improperly. Since $g_j^5 \in L^1$, ρ_j has no atom at zero. Integration by parts against e^{-Et} , followed by dominated convergence in ρ_j , gives

$$R_j(v) = 2 \lim_{T \rightarrow \infty} \int_0^T g_j(t) \sin(vt) dt. \quad (129)$$

The transform defining H is absolutely convergent.

Let $\Xi_j = \rho_j^{*5}$. Then

$$g_j(t)^5 = \frac{1}{32} \int e^{-Et} \Xi_j(dE), \quad S_j(v) = \frac{1}{16} \int \frac{v}{v^2 + E^2} \Xi_j(dE). \quad (130)$$

The L^1 assumption gives $\int E^{-1} \Xi_j(dE) < \infty$, so $H(v)/v$ is integrable near zero. At infinity the unit masses of Ξ_1 and Ξ_2 cancel. Their second moments are finite because

$$\int E^2 \Xi_j(dE) = 5 \int E^2 \rho_j(dE) + 20 \left(\int E \rho_j(dE) \right)^2 < \infty.$$

Using $v/(v^2 + E^2) = v^{-1} - E^2/[v(v^2 + E^2)]$ gives $|H(v)| \leq Cv^{-3}$ for $v \geq 1$. Hence $H, H/v \in L^1(0, \infty)$.

Each g_j decreases from $1/2$ to zero, so f has bounded variation and vanishes at both endpoints. Its truncated sine transforms are bounded by C/v . Fourier inversion for the odd extension of h , followed first by Fubini on $[0, T]$ and then by dominated convergence using $H(v)/v \in L^1$, yields the pairing

$$2 \int_0^\infty (g_1 - g_2)(g_1^5 - g_2^5) dt = \frac{1}{\pi} \int_0^\infty (R_1 - R_2)(S_1 - S_2) dv. \quad (131)$$

The left side is nonnegative. The Dyson equation gives $S_j = R_j^{-1} - v$, and therefore the right integrand is

$$(R_1 - R_2)(R_1^{-1} - R_2^{-1}) = -\frac{(R_1 - R_2)^2}{R_1 R_2} \leq 0. \quad (132)$$

Both sides of (131) vanish. Strict monotonicity of $x \mapsto x^5$ on $[0, \infty)$ gives $g_1 = g_2$ almost everywhere and hence everywhere. Injectivity of the Laplace transform gives $\rho_1 = \rho_2$. $\square$

Corollary 9.5 (Full zero-temperature convergence). *There is a unique pair (ρ_6, g_6) satisfying (6)–(7), and*

$$\rho_\beta \Rightarrow \rho_6, \quad g_\beta \rightarrow g_6 \quad \text{locally uniformly,} \quad \int_0^{\beta/2} g_\beta(t)^6 dt \longrightarrow \int_0^\infty g_6(t)^6 dt. \quad (133)$$

Proof. Every subsequence has a further subsequence covered by Theorem 9.3, and Theorem 9.4 identifies all limits. The last convergence follows from local uniform convergence and the uniform L^6 tail (121). $\square$

10 Zero-temperature pressure slope and variational characterization

Define

$$L_6 := \frac{1}{2\pi} \int_0^\infty \log \left(1 + \frac{S_6(v)}{v} \right) dv. \quad (134)$$

Theorem 10.1 (Sextic Schwinger–Dyson pressure slope). *One has*

$$\lim_{\beta \rightarrow \infty} \frac{p_{\text{SD},6}(\beta)}{\beta} = L_6 - \frac{5}{6} \int_0^\infty g_6(t)^6 dt. \quad (135)$$

Proof. For $v \in \mathcal{V}_\beta$, Lemma 9.1 and the determinant product give

$$\frac{1}{\beta} \log \mathcal{D}(\beta^2 G_\beta^5) = \frac{1}{\beta} \sum_{n \geq 0} \log \left(1 + \frac{S_\beta(v_{\beta,n})}{v_{\beta,n}} \right).$$

The second-moment identity and Jensen’s inequality for $x \mapsto (v^2 + x)^{-1}$ imply

$$\frac{v}{v^2 + 1/16} \leq R_\beta(v) \leq \frac{1}{v}.$$

Since $1 + S_\beta(v)/v = 1/[vR_\beta(v)]$ on the grid,

$$0 \leq \log \left(1 + \frac{S_\beta(v)}{v} \right) \leq \log \left(1 + \frac{1}{16v^2} \right). \quad (136)$$

The right side is integrable on $(0, \infty)$. Corollary 9.5 gives uniform convergence of the summand on compact subsets of $(0, \infty)$. Splitting the midpoint sum at ε and M , comparing the decreasing envelope with its integral at both ends, and then taking $\varepsilon \downarrow 0$, $M \uparrow \infty$, proves

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \mathcal{D}(\beta^2 G_\beta^5) = L_6.$$

Finally, symmetry about $1/2$ and $t = \beta u$ give

$$\frac{1}{\beta} \frac{5\beta^2}{12} \int_0^1 G_\beta(u)^6 du = \frac{5}{6} \int_0^{\beta/2} g_\beta(t)^6 dt.$$

Use (133). $\square$

Recall the cone $\mathcal{X}_6$ and the functional $\mathcal{F}_6$ from (12)–(14). The L^1 condition is equivalent to $\int E^{-1} \nu(dE) < \infty$ for the representing measure. For $\sigma \in \mathcal{X}_6$,

$$S_\sigma(v) = 2 \int \frac{v}{v^2 + E^2} \nu(dE) \geq 0. \quad (137)$$

Thus the logarithmic term is finite: near zero it is $O(1 + |\log v|)$, and at infinity it is $O(v^{-2})$.

Proposition 10.2 (Sextic variational principle). *The function $\sigma_6 = g_6^5$ belongs to $\mathcal{X}_6$ and is the unique maximizer of $\mathcal{F}_6$. Moreover,*

$$L_6 = \frac{7}{6} \int_0^\infty g_6(t)^6 dt, \quad \max_{\sigma \in \mathcal{X}_6} \mathcal{F}_6(\sigma) = \frac{1}{3} \int_0^\infty g_6(t)^6 dt. \quad (138)$$

Proof. The convolution representation

$$g_6(t)^5 = \frac{1}{32} \int e^{-Et} \rho_6^{*5}(dE)$$

shows that σ_6 is a positive Laplace transform. Theorem 9.2 gives $\sigma_6 \in L^1$ and $\sigma_6^{6/5} = g_6^6 \in L^1$.

Let $\mathcal{L}(\sigma)$ denote the logarithmic term in $\mathcal{F}_6$. It is concave because S_σ is linear and $x \mapsto \log(1+x)$ is concave. Its supporting functional at σ_6 is determined by

$$\frac{1}{2\pi} \int_0^\infty S_\sigma(v) R_6(v) dv = \int_0^\infty \sigma(t) g_6(t) dt. \quad (139)$$

Indeed, writing $\sigma(t) = \int e^{-Et} \nu(dE)$ and $g_6(t) = \frac{1}{2} \int e^{-Ft} \rho_6(dF)$, Tonelli's theorem and

$$\int_0^\infty \frac{v^2 dv}{(v^2 + E^2)(v^2 + F^2)} = \frac{\pi}{2(E + F)}$$

show that both sides of (139) equal $\frac{1}{2} \iint (E + F)^{-1} \nu(dE) \rho_6(dF)$. It is finite by Hölder, since $\sigma \in L^{6/5}$ and $g_6 \in L^6$.

Because $R_6(v) = 1/(v + S_{\sigma_6}(v))$, the supporting-line inequality gives

$$\mathcal{L}(\sigma) - \mathcal{L}(\sigma_6) \leq \int_0^\infty g_6(t)(\sigma(t) - \sigma_6(t)) dt. \quad (140)$$

The derivative of $(5/6)x^{6/5}$ at $x = \sigma_6 = g_6^5$ is g_6 . Strict convexity therefore gives

$$\frac{5}{6} \{\sigma^{6/5} - \sigma_6^{6/5}\} \geq g_6(\sigma - \sigma_6),$$

with equality only at $\sigma = \sigma_6$. Combining with (140) proves maximality and uniqueness.

For $a > 0$, define $\sigma_a(t) = a^2 \sigma_6(at)$. It remains in $\mathcal{X}_6$, and

$$S_{\sigma_a}(v) = a S_{\sigma_6}(v/a), \quad \mathcal{L}(\sigma_a) = a L_6, \quad \int \sigma_a^{6/5} = a^{7/5} \int g_6^6.$$

The function $a \mapsto \mathcal{F}_6(\sigma_a)$ has a maximum at $a = 1$. Differentiating there gives $L_6 - (7/6) \int g_6^6 = 0$. Substitution into $\mathcal{F}_6$ gives the second identity in (138). $\square$

Corollary 10.3 (Extensive edge density). *The zero-temperature pressure slope is*

$$e_6 := \lim_{\beta \rightarrow \infty} \frac{p_{\text{SD},6}(\beta)}{\beta} = \frac{1}{3} \int_0^\infty g_6(t)^6 dt = \max_{\sigma \in \mathcal{X}_6} \mathcal{F}_6(\sigma). \quad (141)$$

Proof. Combine Theorem 10.1 and Proposition 10.2. $\square$

11 From pressure to the sextic spectral edge

Proposition 11.1 (Sharp Gaussian Lipschitz bound). *For every even $N \geq 6$ and every $t > 0$,*

$$\mathbb{P}(|\lambda_{\max}(H_{N,6}) - \mathbb{E}\lambda_{\max}(H_{N,6})| \geq t) \leq 2 \exp \left\{ -\frac{(N-5)t^2}{2} \right\}. \quad (142)$$

For the extensive Hamiltonian,

$$\text{Lip}(\lambda_{\max}(\mathcal{H}_{N,6}))^2 \leq \frac{\alpha_{N,6}N}{384(N-5)}. \quad (143)$$

Proof. Fix a five-set $T \subset [N]$. The Hermitian unitaries $\{\Psi_{T \cup \{x\}} : x \notin T\}$ pairwise anticommute, since two supports intersect in the odd number five. For any unit vector v , if $a_x = \langle v, \Psi_{T \cup \{x\}} v \rangle$, then $B = \sum_x a_x \Psi_{T \cup \{x\}}$ satisfies $B^2 = (\sum_x a_x^2)I$. Hence

$$\sum_x a_x^2 = \langle v, Bv \rangle \leq \|B\| = \left(\sum_x a_x^2 \right)^{1/2},$$

so $\sum_x a_x^2 \leq 1$. Summing over all five-sets, each sextet being counted six times, gives

$$6 \sum_{|A|=6} |\langle v, \Psi_A v \rangle|^2 \leq \binom{N}{5}. \quad (144)$$

The variational formula

$$\lambda_{\max}(H_{N,6}(J)) = \sup_{\|v\|=1} \binom{N}{6}^{-1/2} \sum_{|A|=6} J_A \langle v, \Psi_A v \rangle$$

expresses the top eigenvalue as a supremum of linear functions whose Euclidean slope norms, by (144), are at most

$$\left\{ \binom{N}{6}^{-1} \frac{1}{6} \binom{N}{5} \right\}^{1/2} = \frac{1}{\sqrt{N-5}}.$$

Thus the supremum has the same Lipschitz constant. The Gaussian concentration theorem for an L -Lipschitz function [14, Chapter 1] gives $\mathbb{P}(|F - \mathbb{E}F| \geq t) \leq 2e^{-t^2/(2L^2)}$, proving (142). Multiplication by the conversion factor in (5) proves (143). $\square$

Proposition 11.2 (Expected extensive edge). *Let $\tilde{\lambda}_N = \lambda_{\max}(\mathcal{H}_{N,6})$. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \tilde{\lambda}_N = e_6. \quad (145)$$

Proof. The irreducible Clifford representation has dimension $d_N = 2^{N/2}$. For every realization,

$$d_N^{-1} e^{\beta \tilde{\lambda}_N} \leq \text{tr} e^{\beta \mathcal{H}_{N,6}} \leq e^{\beta \tilde{\lambda}_N}.$$

Put $\ell_N = N^{-1} \mathbb{E} \tilde{\lambda}_N$ and $p_{N,6}(\beta) = N^{-1} \log \mathbb{E} \text{tr} e^{\beta \mathcal{H}_{N,6}}$. Jensen's inequality applied to the left inequality gives

$$\ell_N \leq \frac{p_{N,6}(\beta)}{\beta} + \frac{\log 2}{2\beta}. \quad (146)$$

For an L -Lipschitz Gaussian function, the log-Sobolev/Herbst bound is $\log \mathbb{E} e^{s(F - \mathbb{E}F)} \leq s^2 L^2 / 2$. Apply it to the right inequality and use (143):

$$p_{N,6}(\beta) \leq \beta \ell_N + \frac{\beta^2 \alpha_{N,6}}{768(N-5)}. \quad (147)$$

Fix β , let $N \rightarrow \infty$, and invoke Theorem 8.3. Equations (146)–(147) yield

$$\frac{p_{\text{SD},6}(\beta)}{\beta} \leq \liminf_N \ell_N \leq \limsup_N \ell_N \leq \frac{p_{\text{SD},6}(\beta)}{\beta} + \frac{\log 2}{2\beta}.$$

Let $\beta \rightarrow \infty$ and use Corollary 10.3. □

Lemma 11.3 (Exact samplewise sextic spectral symmetry). *For every even N and every real coupling realization,*

$$\text{Spec}(H_{N,6}) = -\text{Spec}(H_{N,6}), \quad \lambda_{\min}(H_{N,6}) = -\lambda_{\max}(H_{N,6}). \quad (148)$$

The same holds for $\mathcal{H}_{N,6}$.

Proof. The complex Clifford algebra on an even number of generators has, up to unitary equivalence, a unique irreducible representation. The entrywise-conjugate matrices $\bar{\mathbf{q}}_i$ form another irreducible representation of the same relations, so there is a unitary U with $U\bar{\mathbf{q}}_i U^* = \mathbf{q}_i$ for every i . If K denotes coordinatewise complex conjugation, the antiunitary $C = UK$ satisfies $C\mathbf{q}_i C^{-1} = \mathbf{q}_i$ and $CiC^{-1} = -i$. Therefore, for a sextet A ,

$$C\Psi_A C^{-1} = C(i^3 Q_A) C^{-1} = (-i)^3 Q_A = -i^3 Q_A = -\Psi_A.$$

All J_A are real, hence $CH_{N,6}C^{-1} = -H_{N,6}$. If $H_{N,6}v = \lambda v$, then $H_{N,6}Cv = -\lambda Cv$, proving (148). □

Proof of Theorem 1.1. By (5) and Proposition 11.2,

$$\begin{aligned} \frac{\mathbb{E}\lambda_{\max}(H_{N,6})}{\sqrt{N}} &= \sqrt{\frac{384}{\alpha_{N,6}}} \frac{\mathbb{E}\lambda_{\max}(\mathcal{H}_{N,6})}{N} \\ &\longrightarrow \sqrt{384} e_6 = \frac{8\sqrt{6}}{3} \int_0^\infty g_6(t)^6 dt = \frac{16}{\sqrt{6}} \int_0^\infty g_6(t)^6 dt = E_6. \end{aligned}$$

For every $\varepsilon > 0$, Proposition 11.1, with $t = \varepsilon\sqrt{N}$, gives a summable bound

$$\mathbb{P}\left(|\lambda_{\max}(H_{N,6}) - \mathbb{E}\lambda_{\max}(H_{N,6})| \geq \varepsilon\sqrt{N}\right) \leq 2e^{-\varepsilon^2 N(N-5)/2}.$$

The first Borel–Cantelli lemma proves almost-sure convergence along even N . Lemma 11.3 gives the bottom edge on the same event. Finally, Proposition 10.2 gives $E_6 = \sqrt{384} \max \mathcal{F}_6 = 8\sqrt{6} \max \mathcal{F}_6$. □

12 Extension to arbitrary fixed even interaction order

Sections 3–11 are the $q = 6$ specialization. This section carries the same argument with q symbolic, recording each arity-dependent constant as a function of q : the Majorana algebra, positive-Lehmann closure, interpolation, deletion and locality estimates, transfer hypotheses, cavity restoration, zero-temperature limit, and edge concentration. Throughout, q is fixed; constants may depend on q but never on N .

For a fixed even integer $q \geq 4$, let

$$\Omega_q = \mathbb{R}^{\binom{N}{q}}, \quad \mathbb{P}_q = \bigotimes_{A \in \binom{[N]}{q}} \mathcal{N}(0, 1),$$

and use the coordinate family $(J_A)_{A \in \binom{[N]}{q}}$ to couple all even system sizes. Define

$$\Psi_A = i^{q/2} \prod_{i \in A} \mathbf{q}_i, \quad H_{N,q} = \binom{N}{q}^{-1/2} \sum_{A \in \binom{[N]}{q}} J_A \Psi_A, \quad (149)$$

with all N coupled by one infinite i.i.d. Gaussian family. Put

$$\alpha_{N,q} = \prod_{j=1}^{q-1} \left(1 - \frac{j}{N}\right), \quad m_q := 2^{2-q}. \quad (150)$$

The corresponding extensive Hamiltonian is

$$\mathcal{H}_{N,q} := 2^{-q/2} \sqrt{\frac{(q-1)!}{N^{q-1}}} \sum_{|A|=q} J_A \Psi_A. \quad (151)$$

A normalized zero-temperature q -Dyson solution is a probability measure ρ_q on $[0, \infty)$ such that, with

$$g_q(t) = \frac{1}{2} \int_0^\infty e^{-Et} \rho_q(dE), \quad (152)$$

$$R_q(v) = \int_0^\infty \frac{v}{v^2 + E^2} \rho_q(dE), \quad S_q(v) = 2 \int_0^\infty \sin(vt) g_q(t)^{q-1} dt, \quad (153)$$

one has

$$\int E^2 \rho_q(dE) = m_q, \quad g_q^{q-1} \in L^1(0, \infty), \quad R_q(v) = \frac{1}{v + S_q(v)} \quad (v > 0). \quad (154)$$

12.1 Statement of the fixed-arity theorem

Theorem 12.1 (Pressure and both spectral edges for every fixed even interaction order). *For every fixed even $q \geq 4$, there is a unique solution of (152)–(154). At every fixed $\beta \geq 0$, the annealed pressure of (151) satisfies*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \operatorname{tr} e^{\beta \mathcal{H}_{N,q}} = p_{\text{SD},q}(\beta) := \log \mathcal{D}(\beta^2 G_{\beta,q}^{q-1}) - \frac{q-1}{2q} \beta^2 \int_0^1 G_{\beta,q}(u)^q du, \quad (155)$$

where $G_{\beta,q} = D(\beta^2 G_{\beta,q}^{q-1})$ is the unique normalized positive-Lehmann solution. As $N \rightarrow \infty$ through even integers,

$$\frac{\lambda_{\max}(H_{N,q})}{\sqrt{N}} \rightarrow E_q := \frac{2^{q/2+1}}{\sqrt{q}} \int_0^\infty g_q(t)^q dt \quad \text{almost surely.} \quad (156)$$

The integral is finite and strictly positive. If $q \equiv 2 \pmod{4}$, then the spectrum is exactly symmetric for every finite N and every realization. If $q \equiv 0 \pmod{4}$, the lower-edge conclusion below holds on a probability-one event by the disorder involution $J \mapsto -J$. In both cases,

$$\frac{\lambda_{\min}(H_{N,q})}{\sqrt{N}} \rightarrow -E_q, \quad \frac{\|H_{N,q}\|}{\sqrt{N}} \rightarrow E_q \quad \text{almost surely.} \quad (157)$$

Equivalently, if

$$\mathcal{X}_q = \left\{ \sigma(t) = \int_0^\infty e^{-Et} \nu(dE) : \nu \geq 0 \text{ finite}, \sigma \in L^1 \cap L^{q/(q-1)} \right\}, \quad (158)$$

$$S_\sigma(v) := 2 \int_0^\infty \sin(vt) \sigma(t) dt, \quad (159)$$

$$\mathcal{F}_q(\sigma) = \frac{1}{2\pi} \int_0^\infty \log \left(1 + \frac{S_\sigma(v)}{v} \right) dv - \frac{q-1}{q} \int_0^\infty \sigma(t)^{q/(q-1)} dt, \quad (160)$$

then $\sigma_q = g_q^{q-1}$ is the unique maximizer and

$$\max_{\sigma \in \mathcal{X}_q} \mathcal{F}_q(\sigma) = \frac{2}{q} \int_0^\infty g_q(t)^q dt, \quad E_q = 2^{q/2} \sqrt{q} \max_{\sigma \in \mathcal{X}_q} \mathcal{F}_q(\sigma). \quad (161)$$

12.2 Algebra, normalization, positive closure, and finite-temperature inputs

Lemma 12.2 (Fixed-arity model-dependent inputs). *For every fixed even $q \geq 4$, the following statements hold with constants depending on q but not on N .*

(Q1) Ψ_A in (149) is a Hermitian unitary and, for even-cardinality sets, $\Psi_A \Psi_B = (-1)^{|A \cap B|} \Psi_B \Psi_A$.
Moreover

$$\mathcal{H}_{N,q} = \sqrt{\frac{\alpha_{N,q} N}{q 2^q}} H_{N,q}. \quad (162)$$

(Q2) If $r = q - 1$ and G_μ is a normalized positive Lehmann kernel, then $G_\mu^r = G_{\Gamma_r(\mu)}$, where

$$\begin{aligned} & \int f(s) \Gamma_r(\mu)(ds) \\ &= \int \mu(dx_1) \cdots \mu(dx_r) \sum_{\varepsilon_2, \dots, \varepsilon_r = \pm 1} \frac{\cosh\left((x_1 + \sum_{j=2}^r \varepsilon_j x_j)/2\right)}{2^{2r-2} \prod_{j=1}^r \cosh(x_j/2)} f\left(\left|x_1 + \sum_{j=2}^r \varepsilon_j x_j\right|\right). \end{aligned} \quad (163)$$

This is a positive measure of mass $m_q = 2^{1-r}$ and depends weakly continuously on μ .

(Q3) Deleting all q -sets meeting a marked set of cardinality k in at least two sites costs, on every fixed normalized contour word,

$$C_W \beta^{2q} \frac{q-1}{2^{q+1}} \frac{k(k-1)}{N}. \quad (164)$$

Enlarging the marked set by r_0 fixed labels costs at most

$$C_W \beta^{2q} \frac{q-1}{2^q N} \left(k r_0 + \binom{r_0}{2} \right). \quad (165)$$

With-replacement sampling of q marked coordinates differs from sampling q distinct coordinates by at most $q(q-1)/(2k)$ for bounded overlap tests.

(Q4) In the unequal-time locality expansion, a support built from s interaction letters has size at most $q(s+2)$. The label counts are

$$\#\{E \in \binom{\Lambda}{q} : E \cap S \neq \emptyset\} \leq C_q(s+1)N^{q-1}, \quad (166)$$

$$\#\{T \in \binom{\Lambda}{q-1} : T \cap S \neq \emptyset\} \leq C_q(s+1)N^{q-2}. \quad (167)$$

Since every Gaussian coefficient square is $O_q(N^{-(q-1)})$, the forced marked block has mass $O_q((s+1)/N)$. Differentiating the whole holomorphic commutator square at total order s leaves at most $s+4$ exterior Gaussian occurrences. Consequently its derivative is bounded by

$$\frac{C_q^{s+1}(s+1)^{s+4}(1+k)^s}{N}, \quad (168)$$

and the Taylor series has a positive radius. More explicitly, the scalar holomorphic commutator square contains four undifferentiated endpoint fields, two in each algebraic commutator. A derivative with respect to either complex time differentiates one Heisenberg evolution and inserts one copy of the generator; after expanding that generator, it adds at most one Gaussian Hamiltonian letter. Deterministic bath and local-hybridization letters are bounded and do not contribute to the Gaussian occurrence count. In the partial-matching integration-by-parts expansion, the first terminal-family block still has to meet the previously exposed support whether it is paired or differentiates the terminal trace word, so the forced N^{-1} incidence factor survives every matching pattern. This yields the majorant $C_q^{s+1}(s+1)^{s+4}(1+k)^s/N$ before division by $s!$, whose consecutive-term ratio tends to $eC_q(1+k)u$. Hence the compact-time commutator theorem and the triangular transfer theorem hold for every fixed q .

(Q5) In the one-site cavity expansion, residual $(q-1)$ -fold blocks attached to disjoint labels are odd, square to $-I$ after the distinguished Majorana is removed, and anticommute. Their pairing sign is $(-1)^{\text{cr}(\mathfrak{p})}$. If

$$\gamma_N^2 = \theta\beta^2 2^{-q} \frac{(q-1)!}{N^{q-1}}, \quad v_N = \gamma_N^2 \binom{N-1}{q-1},$$

then one disjoint chord contributes $v_N(2G)^{q-1}$. Matching it to a quadratic source gives

$$s_N = 2^q v_N = \theta\beta^2 \alpha_{N,q}. \quad (169)$$

Intersections among p residual labels and s test sites have total probability $O_q((p^2 + ps)/N)$, and the factorial Duhamel majorant makes their summed contribution $O_{q,\beta,T,s}(N^{-1})$.

(Q6) The interpolation derivative is

$$P'_M(\theta) = \frac{\beta^2 \alpha_{N,q}}{2q} \int K_q - \frac{1}{2} \int \Sigma_M G, \quad K_q = 2^{-q} f_q. \quad (170)$$

With $h = (\Sigma_M/(\beta^2 \alpha_{N,q}))^{1/(q-1)}$, its defect is

$$\mathcal{B}_q(x, h) = x^q - qh^{q-1}x + (q-1)h^q = (x-h)^2 \sum_{j=0}^{q-2} (j+1)h^j x^{q-2-j} \geq 0. \quad (171)$$

On $0 \leq x, h \leq 1/2$ this factorization also gives the explicit stability bound

$$0 \leq \mathcal{B}_q(x, h) \leq \frac{q(q-1)}{2^{q-1}} (x-h)^2. \quad (172)$$

(Q7) The Gaussian Lipschitz constant of $\lambda_{\max}(H_{N,q})$ is at most $(N-q+1)^{-1/2}$.

Proof. Item (Q1) is Lemmas 3.1–3.2 with q left symbolic. For (Q2), use

$$\prod_{j=1}^r \cosh a_j = 2^{1-r} \sum_{\varepsilon_2, \dots, \varepsilon_r = \pm 1} \cosh \left(a_1 + \sum_{j=2}^r \varepsilon_j a_j \right).$$

After division by the r Lehmann denominators, this gives (163). At thermal time zero its weights sum to $2^{1-r} = m_q$. Positivity and weak continuity follow exactly as in Lemma 4.1.

For (Q3), every deleted q -set contains a marked pair, so its number is at most $\binom{k}{2} \binom{N-2}{q-2}$. Multiplication by the coefficient variance $\beta^2 2^{-q} (q-1)! N^{-(q-1)}$ gives (164). A newly deleted interaction contains either a marked–new pair or a new–new pair, giving (165). The sampling bound is the union bound over the $\binom{q}{2}$ possible collisions.

For (Q4), disjoint even monomials commute, so an admissible interaction label must meet the current support. Summing by the nonempty intersection size gives (166)–(167). Before the first occurrence of the terminal marked family, its marked site is absent; hence its $(q-1)$ -set must meet the preceding support. This supplies one factor $N^{q-2}N^{-(q-1)} = N^{-1}$. All other blocks have bounded conditional mass. The number of exterior Gaussian occurrences is determined by the derivative order and the four endpoint fields, not by q , so it is at most $s+4$. Partial-matching Gaussian integration by parts gives (168); division by $s!$ has consecutive ratio tending to $eC_q(1+k)u$. The fixed-mark modular proof and the slow diagonal in Theorem 6.10 use only this bound, the covariance-mass estimate $\gamma_N^2\binom{N}{q-1} = O_q(1)$, and the deletion estimates. Thus they apply at fixed q .

For (Q5), write $q = 2m$ and, with the distinguished site ordered first, $R_T = \mathbf{q}_0 \Psi_{\{0\} \cup T} = i^m \prod_{t \in T} \mathbf{q}_t$. Since $|T| = 2m-1$,

$$R_T^2 = (-1)^m (-1)^{(2m-1)(m-1)} = -I.$$

Disjoint odd products anticommute. Bringing paired labels together therefore gives precisely one minus sign per chord crossing. Conditional contour factorization supplies $(2G)^{q-1}$, while a quadratic source sG^{q-1} contributes $(s/2)G^{q-1}$; hence $s = 2^q v_N$, and the displayed binomial identity gives (169). Sequential exposure of $(q-1)$ -sets gives the collision probability, and the Wick/simplex estimate is bounded by $C^p/p!$.

For (Q6), Gaussian integration by parts in the q -body couplings produces

$$\frac{\beta^2}{2N} \left(2^{-q} \frac{(q-1)!}{N^{q-1}} \right) \binom{N}{q} 2^q = \frac{\beta^2 \alpha_{N,q}}{2q}.$$

The bath derivative is always $-\frac{1}{2} \int \Sigma_M G$. Adding and subtracting $(q-1)h^q$ gives (171); division by $(x-h)^2$ or direct multiplication proves the factorization.

For (Q7), fix a $(q-1)$ -set T . The family $\{\Psi_{T \cup \{x\}} : x \notin T\}$ anticommutes because any two supports intersect in the odd number $q-1$. The argument of Proposition 11.1 gives

$$q \sum_{|A|=q} |\langle v, \Psi_A v \rangle|^2 \leq \binom{N}{q-1},$$

and hence squared slope at most $\binom{N}{q-1}/(q\binom{N}{q}) = 1/(N-q+1)$. $\square$

12.3 Arity-neutral modular and de Finetti transfer

As in the sextic case, the abstract transfer has two logically distinct limits: first the marked count is fixed and $N \rightarrow \infty$; only afterwards is the marked count allowed to grow along a sufficiently slow diagonal. The following statement preserves that separation.

Proposition 12.3 (Arity-neutral transfer theorem). *Consider one ambient family of finite-dimensional sign systems with a common ordered coordinate sequence, in which the m -marked model is obtained by marking the first m coordinates, so that models with nested marked sets are related by the deletion maps of hypothesis (v) below. Assume as standing structure that every fixed-complexity history is a normalized Gibbs contour word: a product of sector heat propagators with positive heat lengths summing to one and of unitary real-time factors, with all remaining insertions contractions, normalized by the sector partition function; generalized Hölder then bounds each fixed complexity class of such words by one constant, uniformly in the system size and the marked count. Consider the hypotheses:*

- (i) commuting trace-preserving sign involutions with the one-coordinate sector identities;
- (ii) uniformly bounded covariance mass for each marked random field;
- (iii) compact-time asymptotic commutativity of distinct marked fields in all fixed-complexity right-GNS histories;

- (iv) a uniform linear L^2 modulus for the relative cocycles;
- (v) deletion estimates of order $O(k^2/N)$ for a marked set of cardinality k , and additional-deletion estimates of order $O((kr + r^2)/N)$ for r additional moving coordinates, on every normalized fixed-complexity contour word, uniformly in the actual labels, together with exact exchangeability of the deleted models;
- (vi) the two-occurrence contour-incidence condition: the nondistinguished moving coordinates are pairwise distinct, each changes sign exactly twice, and all explicit contractions and local insertions remain in one fixed distinguished subsystem; in particular, no contraction meets a nondistinguished coordinate.

Then the following hold.

- (a) Fixed marked count. If (i)–(iv) hold at one fixed marked count, then, after passage to a subsequence and a Groh–Raynaud ultraproduct, the conclusions of He [4, Proposition 4.13, Proposition 4.16, Theorem 4.17, Proposition 4.18, and Lemma 4.19] hold at that count: the finite sign-sector states have one common support; the compressed matrix cocycles are the Connes cocycles; distinct coordinate algebras commute and are modularly invariant; and every finite-coordinate restriction admits a central product disintegration.
- (b) Slow diagonal. If (i)–(iv) hold at every fixed marked count, with constants allowed to depend on the count, and (v) holds, then there is a deterministic sequence $k_N \uparrow \infty$ with $k_N^2/N \rightarrow 0$ such that, along the k_N -marked deleted models and after passage to a subsequence, the conclusions of He [4, Lemmas 5.2–5.3, Proposition 5.4, and Lemmas 5.5–5.9] hold: one projective directing law on the one-coordinate state space directs every fixed finite family of marked or moving coordinates, uniformly in the actual labels; and the limiting modular data localize fiberwise and extend holomorphically in the open simplex tube and continuously to its closure. Each two-occurrence coordinate of a fixed finite family then contributes one conditional scalar factor $2G_\nu$ as in He [4, Proposition 5.10]. If in addition (vi) holds, the contour words of the stated incidence class factor coordinatewise with antiperiodic factors $\tilde{2G}_\nu$, as in He [4, Corollary 5.12].

The conclusions of part (b) are not available at a fixed marked count: the projective directing law and the contour factorization require the growing diagonal. Neither implication refers to the interaction arity.

Proof. We record, result by result, which hypotheses each of He’s proofs consumes and where; no other model-dependent input occurs in those proofs.

Part (a). The proof of [4, Proposition 4.13] consists of the algebraic Duhamel identities of [4, Lemma 4.12], valid for any sign system satisfying the relations [4, (4.41)–(4.46)] of hypothesis (i), followed by a right-GNS Minkowski estimate whose sole quantitative input is the commutator decay (iii). The proofs of [4, Lemmas 4.14–4.15] define the symmetrized relative-modular spectral measure for two faithful sector states interchanged by a trace-preserving involution from (i), and derive the quadratic tail $\nu\{|x| \geq R\} \leq CR^{-2}$ from the cocycle modulus (iv) and the covariance mass (ii). Given this tightness, the proof of [4, Proposition 4.16] yields equal supports for each adjacent pair of ultraproduct states; connectivity of the sector cube, which is finite because the marked count is fixed, gives one common support — the only step of part (a) that uses the fixed count. [4, Theorem 4.17] is Raynaud’s ultraproduct L^1 -cocycle theorem [9, Proposition 2.2(i)] applied on that support. The proof of [4, Proposition 4.18] compresses the vanishing defects of Proposition 4.13 to the common corner and applies the Connes chain rule, and that of [4, Lemma 4.19] applies Takesaki expectations on the separably acting finite-coordinate algebra; both consume only the conclusions of the previous items.

Part (b). The diagonal is constructed exactly as in the proof of Theorem 6.10, which is stated there for the listed defects and the deletion bound only: enumerate the history moments and the nonnegative defect quantities of part (a) so that the ℓ th item uses only the first ℓ coordinates, as in [4, Lemma 5.3]; each defect vanishes at its own fixed count by (i)–(iv) at that count, and, by the ambient-family compatibility, the ℓ th defect is defined in every model whose marked count is at least ℓ ; and the deletion bound $O(k^2/N)$

of (v) permits increasing thresholds N_r with $d_r(N) \leq r^{-1}$ and $r^2/N \leq r^{-1}$, giving a deterministic $k_N \uparrow \infty$ with $k_N^2/N \rightarrow 0$, as in [4, Proposition 5.4]. Along this diagonal, part (a) holds simultaneously for every fixed finite coordinate family. Exchangeability from (v) and the finite-family product disintegrations of part (a) are exactly the two hypotheses of the projective de Finetti lemma [4, Lemma 5.2], whose proof is abstract: state-space barycenters and a collision fraction $O(r^2/k_N)$, which vanishes because $k_N \rightarrow \infty$. The additional-deletion bound of (v) is the content of [4, Lemma 5.1]; it transfers every conclusion from the marked to the moving coordinates, uniformly in their labels, and back to the undeleted model at total cost $O((k_N r + r^2)/N) = o(1)$. [4, Lemmas 5.5–5.6] are Poisson-kernel estimates in the strip and the simplex tube and consume only the uniform bound on normalized sector words furnished by the standing Gibbs-word normalization via generalized Hölder; [4, Lemmas 5.7–5.9] are statements of Tomita–Takesaki theory (Takesaki expectations, Connes reconstruction inside an invariant subalgebra, and functoriality of spatial derivatives) and localize the modular data fiberwise over the directing law. The proof of [4, Proposition 5.10] deletes additionally on the finite moving family via Lemma 5.1, invokes the single directing law of Proposition 5.4, uses evenness of the background sector word from (i), and produces one factor $2G_\nu(u) = \langle \xi_\nu, \Delta_{\nu\alpha|\nu}^u \xi_\nu \rangle$ per two-occurrence coordinate of an arbitrary finite set; the arity therefore determines only how many such factors are collected downstream, not their form. Finally, [4, Corollary 5.12] assumes precisely the incidence class (vi); its proof applies Proposition 5.10 once per listed coordinate together with the graded KMS sign identity [4, Lemma 5.11], which uses only evenness of the generator and oddness of the coordinate implementers from (i). No step above refers to the interaction arity; the sextic discharge of the same hypothesis list is tabulated in Technical remark 6.11. $\square$

Proposition 12.4 (Verification for fixed-even- q SYK). *For every fixed even $q \geq 4$, the deleted fixed- q SYK sign system satisfies hypotheses (i)–(iv) of Proposition 12.3 at every fixed marked count, together with hypotheses (v)–(vi), uniformly in N and in the interpolation parameter, with constants allowed to depend on q , the fixed inverse temperature, the finite atomic bath, and, in (i)–(iv), on the fixed marked count.*

Proof. The deleted models for all marked counts arise from one SYK Hamiltonian by marking initial segments of the site sequence, and every history in the transfer argument is a normalized Gibbs contour word with heat lengths summing to one, unitary real-time factors, and contractive insertions, so the standing structure holds with the uniform bound given by generalized Schatten Hölder. The sector identities follow from conjugation by the marked Majoranas and the even parity of the q -body generator. The marked-field covariance mass is $2^{-q}(q-1)!N^{-(q-1)}\binom{N}{q-1} = O_q(1)$, independently of the marked count. At each fixed marked count, compact-time asymptotic commutativity and the uniform cocycle modulus follow from the fixed- q derivative estimate and the field-modulus argument above. The deletion and additional-deletion estimates (164)–(165) have the exact scalings $O_q(k^2/N)$ and $O_q((kr + r^2)/N)$ required by (v); exchangeability is exact after deletion. Finally, collision removal leaves pairwise distinct moving coordinates, each occurring exactly twice, and all contractions remain in the distinguished subsystem, which is the incidence class required by He’s Corollary 5.12. $\square$

Corollary 12.5 (Fixed- q modular, de Finetti, and contour transfer). *Fix an even $q \geq 4$, a finite inverse temperature, and a finite atomic bath. In Proposition 6.9 and Theorem 6.10, replace sextets by q -sets, five-sets by $(q-1)$ -sets, and the five conditional coordinate factors in a cavity chord by $q-1$ factors. Then every conclusion (F1)–(F3) and (T1)–(T5) holds, with a deterministic $k_N \uparrow \infty$, $k_N^2/N \rightarrow 0$, and constants allowed to depend on the fixed arity q . In particular, one disjoint cavity chord has conditional bulk factor $(2\tilde{G}_\nu)^{q-1}$.*

Proof. Proposition 12.4 discharges the hypotheses of Proposition 12.3, whose part (a) gives the fixed-mark conclusions and whose part (b) supplies the slow diagonal, the projective directing law, and the contour factorization. It remains to identify these abstract conclusions with the sextic statements (F1)–(F3) and (T1)–(T5) at general even arity.

The sign-system identities use that the interaction generator and each full marked q -body field are even. In the cavity sign calculation, after the distinguished Majorana is stripped off, the residual $(q-1)$ -Majorana block is odd. Both parity statements hold for every even q . For a marked field, the total covariance mass is

$$2^{-q} \frac{(q-1)!}{N^{q-1}} \binom{N}{q-1} = O_q(1),$$

so the field-modulus proof and the relative-modular tightness estimate are unchanged. Item (Q3) of Lemma 12.2 supplies the fixed-mark and additional-deletion estimates. Item (Q4) supplies the sole locality gain, $N^{q-2}N^{-(q-1)} = N^{-1}$, together with the positive-radius derivative majorant (168). These are precisely the model-dependent hypotheses used in Proposition 6.9; all subsequent common-support, modular-invariance, central- disintegration, projective de Finetti, and simplex-tube arguments are statements about the abstract sign system and its coordinate cocycles. The slow-diagonal construction depends only on convergence of each fixed-coordinate defect and on $k_N^2/N \rightarrow 0$, so it also remains unchanged.

For the contour statement, a cavity label has $q-1$ pairwise distinct moving coordinates after collision removal. Each is odd, changes sector exactly twice, and contributes the same one-coordinate factor $2\tilde{G}_\nu$ furnished by Theorem 6.10(T5). Conditional product factorization therefore gives $(2\tilde{G}_\nu)^{q-1}$. No other line of the transfer proof contains the interaction arity. $\square$

12.4 Cavity restoration and deterministic selection

Corollary 12.6 (Fixed-arity cavity restoration). *Fix an even $q \geq 4$, $\beta < \infty$, an atomic positive source Σ_M of mass $m_q\beta^2$, and a sequence $N_j \rightarrow \infty$, $\theta_j \rightarrow \theta \in [0, 1]$. Along the subsequence and directing law π_{cav} furnished by Corollary 12.5, restoring one distinguished physical site has, conditional on the directing state λ , the quadratic source*

$$\Sigma_{\theta, \lambda} = (1 - \theta)\Sigma_M + \theta\beta^2 G_\lambda^{q-1}. \quad (173)$$

The local partition-function ratio is $\mathcal{D}(\Sigma_{\theta, \lambda})$, the local two-state kernel is $D(\Sigma_{\theta, \lambda})$, and the restored directing law is

$$\tilde{\pi}(d\lambda) = \frac{\mathcal{D}(\Sigma_{\theta, \lambda})\pi_{\text{cav}}(d\lambda)}{\int \mathcal{D}(\Sigma_{\theta, \lambda})\pi_{\text{cav}}(d\lambda)}. \quad (174)$$

Moreover,

$$G_\lambda = D\left((1 - \theta)\Sigma_M + \theta\beta^2 G_\lambda^{q-1}\right) \quad \text{for } \tilde{\pi}\text{-almost every } \lambda, \quad (175)$$

and the right side selects the unique deterministic normalized positive-Lehmann solution.

Proof. Write the removed interaction field at system size N_j as

$$X_0^{(j)} = \gamma_j \sum_{T \in \binom{[N_j-1]}{q-1}} g_T \Psi_{\{0\} \cup T}, \quad \gamma_j^2 = \theta_j \beta^2 2^{-q} \frac{(q-1)!}{N_j^{q-1}},$$

and put $v_j = \gamma_j^2 \binom{N_j-1}{q-1}$. At Duhamel order $2p$, Wick's rule, the ordered- simplex volume, and generalized Hölder bound the normalized absolute contribution by

$$\frac{(2p-1)!!}{(2p)!} \{(1+T)^2 v_j\}^p = \frac{\{(1+T)^2 v_j\}^p}{2^p p!}, \quad 0 \leq v_j \leq \frac{\beta^2}{2^q}.$$

Thus the series and all fixed-test variants have a uniform factorial tail. Item (Q5) of Lemma 12.2 bounds assignments in which two residual $(q-1)$ -sets intersect, or one meets the fixed test support, by $O_{q,p,s}(N_j^{-1})$; multiplication by the preceding factorial majorant makes the sum of all collision terms $O_{q,\beta,T,s}(N_j^{-1})$.

For the remaining assignments, all moving bulk coordinates are distinct. Corollary 12.5 gives conditional product factorization, with one factor $2\tilde{G}_\lambda$ per moving coordinate. The residual odd blocks square to $-I$ and anticommute, so bringing equal Wick labels together gives exactly $(-1)^{\text{cr}(\mathfrak{p})}$, the Pfaffian sign of a quadratic Majorana bath. One chord therefore contributes

$$v_j(2G_\lambda)^{q-1}.$$

A quadratic source sG_λ^{q-1} contributes $(s/2)G_\lambda^{q-1}$, so the matching source is

$$s_j = 2^q v_j = \theta_j \beta^2 \prod_{r=1}^{q-1} \left(1 - \frac{r}{N_j}\right) \rightarrow \theta \beta^2.$$

Together with the retained atomic bath, every fixed Duhamel coefficient converges to that of the quadratic source (173); the uniform factorial tail permits removal of the order cutoff. This proves the conditional local kernel and determinant assertions.

The source mass in (173) is $m_q \beta^2$. Positivity of the fermionic determinant product and $\log(1+x) \leq x$ give the uniform bounds

$$1 \leq \mathcal{D}(\Sigma_{\theta,\lambda}) \leq \exp \left\{ m_q \beta^2 \sum_{n \geq 0} \frac{1}{((2n+1)\pi)^2} \right\} = \exp \left\{ \frac{m_q \beta^2}{8} \right\}. \quad (176)$$

Hence (174) is well defined and equivalent to π_{cav} .

It remains to identify the restored local kernel with the directing bulk kernel. Transpose the distinguished site with one unused bulk site and test against finitely many further distinct coordinates at rational thermal times. Finite- N exchangeability, the just-proved coefficientwise cavity limit, collision removal, and the factorial tail give, for rational $u, v_1, \dots, v_s \in (0, 1)$,

$$\int \mathcal{D}(\Sigma_{\theta,\lambda}) \{D(\Sigma_{\theta,\lambda})(u) - G_\lambda(u)\} \prod_{r=1}^s G_\lambda(v_r) \pi_{\text{cav}}(d\lambda) = 0. \quad (177)$$

From here the argument is that of Theorem 7.5, with the fifth-power closure replaced by the $(q-1)$ -fold positive-Lehmann convolution (163). That convolution, source addition, D , and $\mathcal{D}$ are Borel on bounded-mass sets, so Lusin–Souslin [13, Theorem 15.1] makes the signed weight in braces in (177) a Borel function of the rational evaluation vector $\kappa(\lambda) = (G_\lambda(v))_{v \in \mathbb{Q} \cap (0,1)}$, and Stone–Weierstrass forces its pushforward to the closure of $\kappa(\text{supp } \pi_{\text{cav}})$ to vanish. Since the determinant is strictly positive by (176), this gives (175) on the full thermal interval. The Dyson anti-monotonicity identity paired with strict monotonicity of $x \mapsto x^{q-1}$ gives uniqueness of the affine fixed point, so the conditional kernel is deterministic. $\square$

12.5 Finite-temperature pressure, zero temperature, and spectral edges

Proof of Theorem 12.1. By (Q2), the finite-temperature map

$$G \mapsto D(\beta^2 G^{q-1})$$

is a continuous self-map of the compact set of probability measures with second moment at most $m_q \beta^2$. Schauder–Tychonoff gives a fixed point. If G_1, G_2 are two fixed points, the Dyson anti-monotonicity identity gives

$$0 \geq \beta^2 \int_0^1 (G_1 - G_2)(G_1^{q-1} - G_2^{q-1}) du \geq 0,$$

so strict monotonicity of x^{q-1} gives uniqueness. The output second moment is $m_q \beta^2$. The pressure in (155) is

$$p_{\text{SD},q}(\beta) = \log \mathcal{D}(\beta^2 G_{\beta,q}^{q-1}) - \frac{q-1}{2q} \beta^2 \int_0^1 G_{\beta,q}(u)^q du. \quad (178)$$

Fix a finite atomic positive source Σ_M of mass $m_q \beta^2$ converging weakly to $\beta^2 G_{\beta,q}^{q-1}$, and let $G_{M,\theta}^\circ$ be the unique solution of

$$G = D((1-\theta)\Sigma_M + \theta\beta^2 G^{q-1}). \quad (179)$$

Corollary 12.6 gives this equation conditionally under every subsequential directing law and shows that strict monotonicity makes the conditional kernel deterministic. Put

$$h_{M,N} = \left(\frac{\Sigma_M}{\beta^2 \alpha_{N,q}} \right)^{1/(q-1)}, \quad h_M^\circ = \left(\frac{\Sigma_M}{\beta^2} \right)^{1/(q-1)}.$$

For fixed M , define the finite- N interpolation defect

$$\mathcal{R}_{M,N,q}(u, \theta) := K_{q,M,\theta}(u) - q h_{M,N}(u)^{q-1} G_{M,\theta}(u) + (q-1) h_{M,N}(u)^q.$$

Take arbitrary $N_j \rightarrow \infty$ and $\theta_j \in [0, 1]$, then pass to a subsequence with $\theta_j \rightarrow \theta_\infty$. Complex-contour factorization of q distinct coordinate overlaps, together with the collision bound in (Q3), gives

$$\int_0^1 \mathcal{R}_{M,N_j,q}(u, \theta_j) du \longrightarrow \int_0^1 \mathcal{B}_q(G_{M,\theta_\infty}^\circ(u), h_M^\circ(u)) du. \quad (180)$$

The thermal boundary intervals are removed using the uniform bounds $0 \leq G \leq 1/2$ and bounded source mass, exactly as in Proposition 8.1.

The affine fixed-point continuity is uniform in θ : otherwise one could choose $M_j \rightarrow \infty$ and $\theta_j \rightarrow \theta_\infty$ so that a subsequential limit differs from $G_{\beta,q}$, although it solves

$$G = D((1-\theta_\infty)\beta^2 G_{\beta,q}^{q-1} + \theta_\infty \beta^2 G^{q-1}),$$

whose unique solution is $G_{\beta,q}$. Also

$$\|h_M^\circ - G_{\beta,q}\|_{q-1}^{q-1} \leq \beta^{-2} \|\Sigma_M - \beta^2 G_{\beta,q}^{q-1}\|_1 \longrightarrow 0.$$

Hence, by (172),

$$\varepsilon_{M,q} := \sup_{0 \leq \theta \leq 1} \int_0^1 \mathcal{B}_q(G_{M,\theta}^\circ, h_M^\circ) du \longrightarrow 0.$$

If $\limsup_N \sup_\theta |\int \mathcal{R}_{M,N,q}| > \varepsilon_{M,q}$ for a fixed M , a violating sequence and (180) would give a contradiction. Therefore the integrated remainder vanishes in the ordered limits $N \rightarrow \infty$, then $M \rightarrow \infty$. The finite-bath comparison consequently yields

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \operatorname{tr} e^{\beta \mathcal{H}_{N,q}} &= \lim_{M \rightarrow \infty} \left\{ \log \mathcal{D}(\Sigma_M) - \frac{q-1}{2q} \beta^{-2/(q-1)} \int_0^1 \Sigma_M(u)^{q/(q-1)} du \right\} \\ &= p_{\text{SD},q}(\beta). \end{aligned}$$

Thus, for every fixed β ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \operatorname{tr} e^{\beta \mathcal{H}_{N,q}} = p_{\text{SD},q}(\beta), \quad (181)$$

which proves (155). All finite- N errors are quantitative: deletion is $O_q(k_N^2/N)$, additional deletion is $O_q((k_N r + r^2)/N)$, collision diagrams are $O_q(N^{-1})$, and the cavity series has a uniform $C_{q,\beta,T}^p/p!$ tail.

Push the finite-temperature spectral measure forward by $x \mapsto x/\beta$ and write $g_\beta(t) = G_{\beta,q}(t/\beta)$. On the fermionic grid, the dimensionful identities are

$$R_\beta(v) = \frac{1}{v + S_\beta(v)}, \quad S_\beta(v) = 2 \int_0^{\beta/2} \sin(vt) g_\beta(t)^{q-1} dt = m_q R_{\ell,\beta}(v).$$

Choose a grid point with $1 \leq vt \leq 1 + \pi$. The kernel comparison used in Theorem 9.2 yields $R_\beta(v) \geq ct g_\beta(t)$ and $S_\beta(v) \geq ct g_\beta(t)^{q-1}$, hence

$$g_\beta(t)^q \leq C_q t^{-2}, \quad g_\beta(t) \leq C_q t^{-2/q}. \quad (182)$$

Thus g_β^{q-1} and g_β^q have uniform integrable tails. Tightness from the common second moment gives subsequential zero-temperature limits. Weak convergence initially yields only $\int E^2 \rho_q(dE) \leq m_q$. This bound gives a finite first moment and therefore an absolutely continuous, decreasing Laplace transform g . Since $g^{q-1} \in L^1$, the limiting measure has no atom at zero, so $g(t) \downarrow 0$. For $h = g^{q-1}$ one has

$$h(0) = 2^{1-q}, \quad h(t) \downarrow 0, \quad \int_0^\infty |h'(t)| dt = h(0).$$

Integration by parts in $S(v) = 2 \int_0^\infty \sin(vt) h(t) dt$ and the Riemann–Lebesgue lemma give

$$vS(v) = 2h(0) + 2 \int_0^\infty \cos(vt) h'(t) dt \longrightarrow 2^{2-q} = m_q.$$

Hence the reciprocal Dyson equation yields

$$R(v) = \frac{1}{v} - \frac{m_q}{v^3} + o(v^{-3}).$$

On the other hand,

$$v^3 \left(\frac{1}{v} - R(v) \right) = \int_0^\infty \frac{v^2 E^2}{v^2 + E^2} \rho_q(dE) \uparrow \int_0^\infty E^2 \rho_q(dE)$$

by monotone convergence. Comparison recovers the equality $\int E^2 \rho_q(dE) = m_q$.

Uniqueness is the proof of Theorem 9.4 with g^5 replaced by g^{q-1} , the constants $1/32$ and $1/16$ by 2^{1-q} and m_q , and $x \mapsto x^5$ by $x \mapsto x^{q-1}$. The only arity-dependent input is the second moment of the $(q-1)$ -fold convolution of a normalized solution ρ ,

$$\int E^2 \rho^{*(q-1)}(dE) = (q-1) \int E^2 \rho(dE) + (q-1)(q-2) \left(\int E \rho(dE) \right)^2, \quad (183)$$

which is finite for every q and supplies the $O_q(v^{-3})$ decay of $S_1 - S_2$ needed for the Parseval pairing.

The determinant midpoint sum has the integrable envelope $\log(1 + m_q/v^2)$, so

$$\lim_{\beta \rightarrow \infty} \frac{p_{\text{SD},q}(\beta)}{\beta} = L_q - \frac{q-1}{q} \int_0^\infty g_q(t)^q dt, \quad L_q = \frac{1}{2\pi} \int_0^\infty \log \left(1 + \frac{S_q(v)}{v} \right) dv.$$

For $\sigma \in \mathcal{X}_q$, positivity of its Laplace measure gives $S_\sigma \geq 0$ and makes the logarithmic term finite. Tonelli's theorem and $\int_0^\infty v^2 [(v^2 + E^2)(v^2 + F^2)]^{-1} dv = \pi/[2(E + F)]$ give the supporting identity

$$\frac{1}{2\pi} \int_0^\infty S_\sigma(v) R_q(v) dv = \int_0^\infty \sigma(t) g_q(t) dt. \quad (184)$$

The convolution representation $g_q(t)^{q-1} = 2^{1-q} \int e^{-Et} \rho_q^{*(q-1)}(dE)$ and the tail (182) show that $\sigma_q := g_q^{q-1}$ belongs to $\mathcal{X}_q$. Because $R_q = (v + S_{\sigma_q})^{-1}$, concavity of the logarithmic term and the tangent inequality

for the strictly convex function $\frac{q-1}{q}x^{q/(q-1)}$ show that σ_q is the unique maximizer of $\mathcal{F}_q$. Under $\sigma_a(t) = a^2\sigma_q(at)$, the two terms scale as aL_q and $a^{(q+1)/(q-1)}(q-1)\int g_q^q/q$. Differentiation at the maximizing value $a = 1$ gives $L_q = (q+1)\int g_q^q/q$, hence

$$e_q := \lim_{\beta \rightarrow \infty} \frac{\text{PSD},q(\beta)}{\beta} = \frac{2}{q} \int_0^\infty g_q(t)^q dt = \max \mathcal{F}_q. \quad (185)$$

Item (Q7), Gaussian concentration, and the entropy/Herbst sandwich imply $N^{-1}\mathbb{E}\lambda_{\max}(\mathcal{H}_{N,q}) \rightarrow e_q$ and almost-sure concentration about the mean. Using (162) gives

$$\frac{\mathbb{E}\lambda_{\max}(H_{N,q})}{\sqrt{N}} \longrightarrow \sqrt{q2^q} e_q = \frac{2^{q/2+1}}{\sqrt{q}} \int g_q^q,$$

and Borel–Cantelli proves (156). Since $g_q(0) = 1/2$ and g_q is continuous, this integral, and hence E_q , is strictly positive.

Finally, the antiunitary fixing all $\mathbf{q}_i$ sends Ψ_A to $(-1)^{q/2}\Psi_A$. This gives exact spectral symmetry when $q \equiv 2 \pmod{4}$. When $q \equiv 0 \pmod{4}$, let $\mathcal{E}$ be the probability-one event of upper-edge convergence. The image of $\mathcal{E}$ under the measure-preserving involution $J \mapsto -J$ also has probability one. On their intersection, $\lambda_{\min}(H(J)) = -\lambda_{\max}(H(-J))$, proving the lower-edge part of (157). The spectral-norm conclusion follows from $\|H\| = \max\{\lambda_{\max}(H), -\lambda_{\min}(H)\}$. $\square$

Corollary 12.7 (Simultaneous fixed-arity convergence). *Place independent Gaussian families for all even $q \geq 4$ on the countable product probability space $\Omega_* = \prod_{q \in 2\mathbb{N}, q \geq 4} \Omega_q$. There is one event of probability one on which, simultaneously for every fixed even $q \geq 4$, the upper edge, lower edge, and spectral norm converge as in (156)–(157).*

Proof. For each fixed even q , Theorem 12.1 supplies a probability-one event. Their intersection is still probability one because the set of even arities is countable. $\square$

Remark 12.8 (The sextic specialization). *The sextic sections give a detailed specialization of the fixed- q proof, displaying the phase $-i$, the mass $1/16$, the normalization 384, the cavity factor 64, the six-flip partition coefficients, and the sextic Bregman polynomial explicitly.*

13 Large- q asymptotics of the deterministic edge constants

Theorem 12.1 identifies the edge constant E_q for every fixed even interaction order, but the dependence on q is implicit through a nonlinear zero-temperature Schwinger–Dyson equation. In this section we analyze the variational characterization (161) as $q \rightarrow \infty$. The limit is sequential: $N \rightarrow \infty$ is taken first at each fixed q , and only then is q sent to infinity.

Put

$$p_q := \frac{q}{q-1}, \quad \mathcal{J}_q := \sqrt{q2^{1-q}}. \quad (186)$$

The quantity $\mathcal{J}_q$ is the conventional large- q effective coupling for the extensive Hamiltonian (151) with microscopic coupling equal to one. Let

$$e_q := \max_{\sigma \in \mathcal{X}_q} \mathcal{F}_q(\sigma) = \frac{E_q}{2^{q/2}\sqrt{q}}$$

be the extensive edge density from (161).

13.1 Rescaling and a finite- q primal–dual sandwich

For a positive-Laplace function

$$f(x) = \int_0^\infty e^{-sx} \mu(ds), \quad \mu \geq 0 \text{ finite},$$

define

$$H_f(y) := \frac{1}{y} \int_0^\infty \sin(yx) f(x) dx = \int_0^\infty \frac{\mu(ds)}{s^2 + y^2}, \quad y > 0. \quad (187)$$

The second formula follows from Fubini; the L^1 condition imposed below gives the needed absolute integrability. Let

$$\mathcal{Y}_q := \{f : f \text{ is positive Laplace and } f \in L^1(0, \infty) \cap L^{p_q}(0, \infty)\}.$$

For $f \in \mathcal{Y}_q$, set

$$\sigma_f(t) := 2^{1-q} f(\mathcal{J}_q t). \quad (188)$$

This is a bijection from $\mathcal{Y}_q$ onto $\mathcal{X}_q$.

Lemma 13.1 (Large- q rescaling). *For $f \in \mathcal{Y}_q$,*

$$\Phi_q(f) := \frac{q^2}{\mathcal{J}_q} \mathcal{F}_q(\sigma_f) = \frac{q^2}{2\pi} \int_0^\infty \log \left(1 + \frac{2H_f(y)}{q} \right) dy - \frac{q-1}{2} \int_0^\infty f(x)^{p_q} dx. \quad (189)$$

Consequently,

$$M_q := \max_{f \in \mathcal{Y}_q} \Phi_q(f) = \frac{q^2}{\mathcal{J}_q} e_q = \frac{qE_q}{\sqrt{2}}. \quad (190)$$

Proof. At $v = \mathcal{J}_q y$, equations (187)–(188) give

$$\frac{S_{\sigma_f}(v)}{v} = \frac{2^{2-q}}{\mathcal{J}_q^2} H_f(y) = \frac{2}{q} H_f(y).$$

Moreover,

$$\int_0^\infty \sigma_f(t)^{p_q} dt = \frac{2^{-q}}{\mathcal{J}_q} \int_0^\infty f(x)^{p_q} dx.$$

Substitution into the definition of $\mathcal{F}_q$, followed by multiplication by $q^2/\mathcal{J}_q$, proves (189). The last identity follows from $E_q = 2^{q/2} \sqrt{q} e_q$ and $\mathcal{J}_q^2 = q2^{1-q}$. $\square$

The following dual inequality is useful both for the asymptotic theorem and as a nonasymptotic certificate. Let

$$h(x) = \int_0^\infty e^{-sx} \eta(ds)$$

be the Laplace transform of a probability measure and assume $h^q \in L^1(0, \infty)$. Define

$$r_h(y) := \int_0^\infty \frac{y^2}{y^2 + s^2} \eta(ds), \quad 0 < r_h(y) \leq 1. \quad (191)$$

The representation in (191), rather than an L^2 Fourier formula, is the definition; this matters because the certificate used below need not lie in L^2 .

Proposition 13.2 (Exact primal–dual inequality). *For every normalized positive-Laplace h with $h^q \in L^1$,*

$$M_q \leq \mathcal{U}_q(h) := \frac{q^2}{2\pi} \int_0^\infty (r_h(y) - 1 - \log r_h(y)) \, dy + \frac{1}{2} \int_0^\infty h(x)^q \, dx, \quad (192)$$

where the right side is allowed to be $+\infty$.

Proof. Fix $f \in \mathcal{Y}_q$ and put $z(y) = 2H_f(y)/q$. The tangent line to the concave function $z \mapsto \log(1+z)$ at $z_h = r_h^{-1} - 1$ gives

$$\log(1+z) \leq r_h - 1 - \log r_h + r_h z. \quad (193)$$

Write $f(x) = \int e^{-ax} \nu(da)$. Since $f \in L^1$, one has $\int a^{-1} \nu(da) < \infty$. All integrands in the next calculation are nonnegative, so Tonelli's theorem and the elementary two-denominator integral yield

$$\begin{aligned} \int_0^\infty r_h(y) H_f(y) \, dy &= \iint \eta(ds) \nu(da) \int_0^\infty \frac{y^2 \, dy}{(y^2 + s^2)(y^2 + a^2)} \\ &= \frac{\pi}{2} \iint \frac{\eta(ds) \nu(da)}{a + s} = \frac{\pi}{2} \int_0^\infty h(x) f(x) \, dx. \end{aligned} \quad (194)$$

Substituting (193) and (194) into (189) gives

$$\Phi_q(f) \leq \frac{q^2}{2\pi} \int (r_h - 1 - \log r_h) + \int_0^\infty \left(\frac{q}{2} h f - \frac{q-1}{2} f^{q/(q-1)} \right) \, dx.$$

For each x ,

$$\sup_{u \geq 0} \left\{ \frac{q}{2} h(x) u - \frac{q-1}{2} u^{q/(q-1)} \right\} = \frac{1}{2} h(x)^q,$$

with optimizer $u = h(x)^{q-1}$. Taking the supremum over f proves the claim. $\square$

The limiting primal profile and its natural Gamma dual deformation are

$$f_*(x) := \frac{1}{(1+x)^2}, \quad h_q(x) := (1+x)^{-2/q}. \quad (195)$$

Both are positive Laplace transforms:

$$f_*(x) = \int_0^\infty s e^{-s} e^{-sx} \, ds, \quad h_q(x) = \frac{1}{\Gamma(2/q)} \int_0^\infty s^{2/q-1} e^{-s} e^{-sx} \, ds. \quad (196)$$

Let $H_* := H_{f_*}$ and

$$r_q(y) := \frac{1}{\Gamma(2/q)} \int_0^\infty \frac{y^2}{y^2 + s^2} s^{2/q-1} e^{-s} \, ds. \quad (197)$$

Since $\int h_q^q = \int_0^\infty (1+x)^{-2} \, dx = 1$, Lemma 13.1 and Proposition 13.2 give the following exact bounds.

Theorem 13.3 (Explicit finite- q edge certificates). *For every even $q \geq 4$,*

$$\frac{q^2}{2\pi} \int_0^\infty \log \left(1 + \frac{2H_*(y)}{q} \right) \, dy - \frac{(q-1)^2}{2(q+1)} \leq \frac{qE_q}{\sqrt{2}} \quad (198)$$

$$\leq \frac{q^2}{2\pi} \int_0^\infty (r_q(y) - 1 - \log r_q(y)) \, dy + \frac{1}{2}. \quad (199)$$

Both integrals are finite.

Proof. The lower bound is $\Phi_q(f_*) \leq M_q$ and $\int f_*^{p_q} = (q-1)/(q+1)$. The upper bound is Proposition 13.2 with $h = h_q$. Finiteness of the lower integral follows from the logarithmic behavior of H_* at zero and $H_*(y) = O(y^{-2})$ at infinity. For the upper integral, write $\alpha = 2/q$. Restricting the Gamma integral to $0 < s < y$ gives $r_q(y) \geq c_\alpha y^\alpha$ for $0 < y \leq 1$, while $1 - r_q(y) \leq C_\alpha y^{-2}$ for $y \geq 1$. Hence $r_q - 1 - \log r_q = O_\alpha(1 + |\log y|)$ at zero and $O_\alpha(y^{-4})$ at infinity. This proves finiteness for every $q \geq 4$; the uniform estimates needed for the asymptotic expansion are supplied below. $\square$

13.2 The limiting profile and three transform constants

For a fixed sufficiently regular positive-Laplace function, expansion of (189) suggests the limiting functional

$$\Phi_\infty(f) := \frac{1}{2} \int_0^\infty (f - f \log f) \, dx - \frac{1}{\pi} \int_0^\infty H_f(y)^2 \, dy. \quad (200)$$

The identity

$$\frac{1}{\pi} \int_0^\infty H_f(y)^2 \, dy = \frac{1}{2} \iint_{(0,\infty)^2} \min(x, z) f(x) f(z) \, dx \, dz \quad (201)$$

follows from the sine-kernel formula.

Proposition 13.4 (Limiting optimizer and gap identity). *Let $\mathcal{A}_\infty$ be the class of measurable positive-Laplace functions $f \geq 0$ such that*

$$\int_0^\infty f(x) \, dx < \infty, \quad \int_0^\infty f(x) |\log f(x)| \, dx < \infty, \quad \int_0^\infty \left(\int_r^\infty |f(x) - f_*(x)| \, dx \right)^2 \, dr < \infty.$$

For $f \in \mathcal{A}_\infty$, both terms in (200) are finite, the relative-entropy term below is well defined, and the min-kernel quadratic form is finite. On this class the unique maximizer is $f_(x) = (1+x)^{-2}$ and $\Phi_\infty(f_*) = 1$. More precisely, with $u = f - f_*$ and the convention $0 \log 0 = 0$,*

$$\begin{aligned} \Phi_\infty(f_*) - \Phi_\infty(f) &= \frac{1}{2} \int_0^\infty \left[f \log \left(\frac{f}{f_*} \right) - f + f_* \right] \, dx \\ &\quad + \frac{1}{2} \iint_{(0,\infty)^2} \min(x, z) u(x) u(z) \, dx \, dz. \end{aligned} \quad (202)$$

Both terms on the right are nonnegative and vanish simultaneously only when $f = f_$ almost everywhere.*

Proof. We first establish the finiteness assertions. Write $u = f - f_*$ and $F_\phi(r) = \int_r^\infty \phi(x) \, dx$. Since $\log(1+x) = \int_0^x (1+r)^{-1} \, dr$, Tonelli gives

$$\int_0^\infty f(x) \log(1+x) \, dx = \int_0^\infty \frac{F_f(r)}{1+r} \, dr,$$

while $F_{f_*}(r) = (1+r)^{-1}$ gives the pointwise bound $F_f \leq F_{|u|} + (1+r)^{-1}$. The third defining condition of $\mathcal{A}_\infty$ states that $F_{|u|} \in L^2(0, \infty)$, and $(1+r)^{-1} \in L^2(0, \infty)$, so Cauchy–Schwarz yields $\int_0^\infty f(x) \log(1+x) \, dx < \infty$. Since $-\log f_* = 2 \log(1+x)$, this bound and the assumption $\int f |\log f| < \infty$ make the entropy term of (200) and the pointwise nonnegative relative-entropy integrand $f \log(f/f_*) - f + f_*$ absolutely integrable. Likewise, the indicator representation of $\min(x, z)$ displayed below gives, by Tonelli, $\iint \min(x, z) \phi(x) \phi(z) \, dx \, dz = \int_0^\infty F_\phi(r)^2 \, dr$ for every measurable $\phi \geq 0$; taking $\phi = f$ and $\phi = |u|$, both of whose tail integrals lie in $L^2(0, \infty)$ by the bounds above, shows that the min-kernel quadratic form, the second term of (200) (via (201)), and the quadratic and cross terms of the expansion below are all finite. Every rearrangement used in the remainder of the proof therefore converges absolutely.

Let $(Kf)(x) = \int_0^\infty \min(x, z) f(z) \, dz$. Direct integration gives $Kf_*(x) = \log(1+x) = -\frac{1}{2} \log f_*(x)$. Expanding (200) around f_* and using this identity yields (202). The first term is the nonnegative relative-entropy Bregman divergence. For the second, the representation

$$\min(x, z) = \int_0^\infty \mathbf{1}_{\{r \leq x\}} \mathbf{1}_{\{r \leq z\}} \, dr$$

shows that

$$\iint \min(x, z) u(x) u(z) \, dx \, dz = \int_0^\infty \left(\int_r^\infty u(x) \, dx \right)^2 \, dr \geq 0.$$

Finally, $\int f_* = 1$, $\int f_* \log f_* = -2$, and $\pi^{-1} \int H_*^2 = 1/2$, so $\Phi_\infty(f_*) = 1$. $\square$

This limiting calculation identifies the profiles used in the certificates; the proof of the finite- q asymptotic value below relies on the exact inequalities of Theorem 13.3.

Integration by parts and (196) give

$$H_*(y) = \int_0^\infty \frac{\cos(yx)}{1+x} dx = \int_0^\infty \frac{se^{-s}}{s^2+y^2} ds. \quad (203)$$

In particular, $H_* \in L^r(0, \infty)$ for every finite r .

Lemma 13.5 (Exact transform constants). *One has*

$$\int_0^\infty H_*(y) dy = \frac{\pi}{2}, \quad \int_0^\infty H_*(y)^2 dy = \frac{\pi}{2}, \quad \int_0^\infty H_*(y)^3 dy = \frac{\pi^3}{8}. \quad (204)$$

Proof. The first identity follows from (203) and $\int_0^\infty (s^2+y^2)^{-1} dy = \pi/(2s)$. For the second, Tonelli and

$$\int_0^\infty \frac{dy}{(y^2+a^2)(y^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

give

$$\int H_*^2 = \frac{\pi}{2} \iint \frac{e^{-(a+b)}}{a+b} da db = \frac{\pi}{2}.$$

For the third, use

$$\int_0^\infty \frac{dy}{(y^2+a^2)(y^2+b^2)(y^2+c^2)} = \frac{\pi(a+b+c)}{2abc(a+b)(b+c)(c+a)}.$$

After scaling a, b, c by their sum, the remaining simplex integral is

$$\int_{\substack{x>0, z>0 \\ x+z<1}} \frac{dx dz}{(x+z)(1-x)(1-z)} = -2 \int_0^1 \frac{\log x}{1-x^2} dx = \frac{\pi^2}{4}.$$

Multiplication by the outer factor $\pi/2$ proves the last identity. $\square$

13.3 The primal lower asymptotic

Proposition 13.6 (Primal asymptotic). *As $q \rightarrow \infty$ through even integers,*

$$M_q \geq 1 + \frac{\pi^2/6 - 2}{q} + O(q^{-2}). \quad (205)$$

Proof. Use f_* in (189). Its nonlinear integral is exact:

$$\int_0^\infty f_*(x)^{p_q} dx = \frac{q-1}{q+1}. \quad (206)$$

For all $z \geq 0$,

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \leq \log(1+z) \leq z - \frac{z^2}{2} + \frac{z^3}{3}. \quad (207)$$

Apply this with $z = 2H_*/q$. Since $H_* \in L^4$, Lemma 13.5 gives

$$\frac{q^2}{2\pi} \int_0^\infty \log \left(1 + \frac{2H_*}{q} \right) dy = \frac{q}{2} - \frac{1}{2} + \frac{\pi^2}{6q} + O(q^{-2}).$$

Meanwhile,

$$\frac{q-1}{2} \int f_*^{p_q} = \frac{(q-1)^2}{2(q+1)} = \frac{q}{2} - \frac{3}{2} + \frac{2}{q} + O(q^{-2}).$$

Their difference is the claimed bound. $\square$

13.4 The Gamma dual certificate and its integrated expansion

Set

$$\alpha := \frac{2}{q}, \quad h_\alpha(x) := (1+x)^{-\alpha}. \quad (208)$$

Then h_α is the Laplace transform of the Gamma($\alpha, 1$) law and $\int h_\alpha^q = 1$. Write

$$r_\alpha(y) := \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^2}{y^2 + s^2} s^{\alpha-1} e^{-s} ds, \quad (209)$$

$$\delta_\alpha(y) := 1 - r_\alpha(y) = \alpha D_\alpha(y), \quad (210)$$

$$D_\alpha(y) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{s^{\alpha+1} e^{-s}}{s^2 + y^2} ds. \quad (211)$$

Lemma 13.7 (Uniform envelope and moment expansions). *There is an absolute constant C such that, for $0 < \alpha \leq 1/4$,*

$$D_\alpha(y) \leq \begin{cases} C(1 + |\log y|), & 0 < y \leq 1, \\ Cy^{-2}, & y \geq 1. \end{cases} \quad (212)$$

Consequently $\sup_{0 < \alpha \leq 1/4} \|D_\alpha\|_{L^4} < \infty$. Moreover, the quadratic moment is exact:

$$\int_0^\infty D_\alpha(y)^2 dy = \frac{\pi}{2(1+2\alpha)} = \frac{\pi}{2} - \pi\alpha + O(\alpha^2), \quad (213)$$

$$\int_0^\infty D_\alpha(y)^3 dy = \frac{\pi^3}{8} + O(\alpha). \quad (214)$$

Proof. For $y \leq 1$, split the s -integral in (211) at $s = y$ and $s = 1$; use $(1 - y^\alpha)/\alpha \leq |\log y|$. For $y \geq 1$, use $(s^2 + y^2)^{-1} \leq y^{-2}$. This proves (212) and the uniform L^4 bound.

Put

$$w_\alpha(s) := \frac{s^\alpha}{\Gamma(1+\alpha)}.$$

Taylor's theorem, uniformly for $0 < \alpha \leq 1/4$, gives

$$w_\alpha(s) = 1 + \alpha(\log s + \gamma_E) + \alpha^2 R_\alpha(s), \quad (215)$$

where

$$|R_\alpha(s)| \leq C(1 + |\log s|^2)(s^{1/4} + s^{-1/4}).$$

The two-denominator kernel gives

$$\int D_\alpha^2 = \frac{\pi}{2} \iint_{(0,\infty)^2} \frac{e^{-(a+b)} w_\alpha(a) w_\alpha(b)}{a+b} da db. \quad (216)$$

The change of variables $s = a + b$ and $t = a/(a + b)$ evaluates (216) exactly:

$$\begin{aligned} \int D_\alpha^2 &= \frac{\pi}{2\Gamma(1+\alpha)^2} \int_0^\infty e^{-s} s^{2\alpha} ds \int_0^1 t^\alpha (1-t)^\alpha dt \\ &= \frac{\pi}{2} \frac{\Gamma(1+2\alpha)}{\Gamma(2+2\alpha)} = \frac{\pi}{2(1+2\alpha)}. \end{aligned}$$

This gives the first line of (213), including its stated expansion.

Similarly, the three-denominator kernel and the substitution $a = rx$, $b = rz$, $c = r(1 - x - z)$ give

$$\int_0^\infty D_\alpha(y)^3 dy = \frac{\pi\Gamma(1+3\alpha)}{2\Gamma(1+\alpha)^3} A(\alpha), \quad (217)$$

where

$$A(\alpha) := \int_{\substack{x>0, z>0 \\ x+z<1}} \frac{[xz(1-x-z)]^\alpha}{(x+z)(1-x)(1-z)} dx dz. \quad (218)$$

The function $|\log(xz(1-x-z))|/[(x+z)(1-x)(1-z)]$ is integrable on the simplex. Hence dominated convergence, applied also to the difference quotient, gives $A(\alpha) = A(0) + O(\alpha)$. Lemma 13.5 computes $A(0) = \pi^2/4$, while $\Gamma(1+3\alpha)/\Gamma(1+\alpha)^3 = 1 + O(\alpha)$. Therefore (217) equals $\pi^3/8 + O(\alpha)$. $\square$

Lemma 13.8 (Integrated dual expansion). *As $\alpha \downarrow 0$,*

$$\int_0^\infty (r_\alpha(y) - 1 - \log r_\alpha(y)) dy = \frac{\pi\alpha^2}{4} + \alpha^3 \left(-\frac{\pi}{2} + \frac{\pi^3}{24} \right) + O(\alpha^4). \quad (219)$$

Proof. A pointwise Taylor expansion is not uniform at $y = 0$, where $r_\alpha(y) \downarrow 0$. Let $y_\alpha = e^{-\alpha^{-1/2}}$. On $[y_\alpha, \infty)$, (212) gives $\delta_\alpha = \alpha D_\alpha \leq C\sqrt{\alpha} < 1/2$ for small α . Therefore, with $\phi(r) = r - 1 - \log r$,

$$\phi(1 - \delta) = \frac{\delta^2}{2} + \frac{\delta^3}{3} + O(\delta^4),$$

and the integrated remainder is $O(\alpha^4)$ by the uniform L^4 bound.

On $(0, y_\alpha)$, restrict the integral in (209) to $0 < s < y$. Since $y \leq 1$,

$$r_\alpha(y) \geq \frac{e^{-1}}{2\Gamma(1+\alpha)} y^\alpha.$$

Hence

$$0 \leq \phi(r_\alpha(y)) \leq -\log r_\alpha(y) \leq C + \alpha |\log y|.$$

Its integral over $(0, y_\alpha)$ is smaller than every power of α . The envelope (212) gives the same conclusion for the omitted D_α^2 and D_α^3 contributions. Thus

$$\int \phi(r_\alpha) = \frac{\alpha^2}{2} \int D_\alpha^2 + \frac{\alpha^3}{3} \int D_\alpha^3 + O(\alpha^4).$$

Substitution of (213)–(214) proves (219). $\square$

Proposition 13.9 (Dual asymptotic). *As $q \rightarrow \infty$ through even integers,*

$$M_q \leq 1 + \frac{\pi^2/6 - 2}{q} + O(q^{-2}). \quad (220)$$

Proof. Apply Proposition 13.2 with $h = h_\alpha$ and $\alpha = 2/q$. Since $q^2/(2\pi) = 2/(\pi\alpha^2)$ and $\int h_\alpha^q = 1$, Lemma 13.8 gives

$$\mathcal{U}_q(h_\alpha) = 1 + \alpha \left(\frac{\pi^2}{12} - 1 \right) + O(\alpha^2) = 1 + \frac{\pi^2/6 - 2}{q} + O(q^{-2}).$$

$\square$

13.5 The large- q edge theorem

Theorem 13.10 (Large- q spectral-edge asymptotics). *For the deterministic fixed-arity edge constants of Theorem 12.1,*

$$\frac{qE_q}{\sqrt{2}} = 1 - \frac{12 - \pi^2}{6q} + O(q^{-2}) \quad (q \rightarrow \infty, q \text{ even}). \quad (221)$$

Equivalently,

$$E_q = \frac{\sqrt{2}}{q} \left(1 - \frac{12 - \pi^2}{6q} + O(q^{-2}) \right). \quad (222)$$

In the extensive normalization,

$$e_q = \frac{\mathcal{J}_q}{q^2} \left(1 - \frac{12 - \pi^2}{6q} + O(q^{-2}) \right), \quad \mathcal{J}_q^2 = q2^{1-q}. \quad (223)$$

Proof. Propositions 13.6 and 13.9 give matching bounds for M_q . Lemma 13.1 converts M_q to E_q and e_q . $\square$

Corollary 13.11 (Sequential N -then- q limit). *On the simultaneous probability-one event of Corollary 12.7,*

$$\lim_{\substack{q \rightarrow \infty \\ q \text{ even}}} \left(\lim_{N \rightarrow \infty} \frac{q\lambda_{\max}(H_{N,q})}{\sqrt{2N}} \right) = 1. \quad (224)$$

The inner limit has first correction $-(12 - \pi^2)/(6q)$.

Remark 13.12 (No joint or double-scaled conclusion). *The proof takes $N \rightarrow \infty$ at each fixed q before $q \rightarrow \infty$. The fixed- q locality, modular-transfer, and cavity estimates in the preceding sections are not controlled uniformly in q . Consequently, neither (222) nor (224) justifies replacing q by $q(N)$; in particular, this is not a theorem in the double-scaled regime $q^2/N \rightarrow \lambda$.*

To compare conventions, write the conventional Majoranas as $\{\chi_i, \chi_j\} = \delta_{ij}$ and

$$\mathcal{H}_{N,q} = i^{q/2} \sqrt{\frac{(q-1)!}{N^{q-1}}} \sum_{i_1 < \dots < i_q} J_{i_1 \dots i_q} \chi_{i_1} \cdots \chi_{i_q}.$$

Thus the microscopic coupling in Tarnopolsky's convention is $J = 1$ and the effective large- q coupling is $\mathcal{J}_q^2 = q2^{1-q}$. Our e_q is the upper-edge density selected by $\exp(+\beta\mathcal{H})$, while Tarnopolsky's conventional ground-state energy density is $-e_q$. Accordingly, his formal magnitude $|E_0|/N$ converts directly to e_q , and the standardized edge is $E_q = 2^{q/2} \sqrt{q} e_q$.

The coefficient in (223) agrees with the formal strong-coupling large- q ground-state expansion of Tarnopolsky [7, Eq. (24)]. The derivation here is different: it uses exact primal and dual inequalities for the deterministic variational problem and does not assume an asymptotic expansion of the unknown Schwinger–Dyson solution. Earlier rigorous work proved the order-sharp lower scaling $\mathbb{E}\lambda_{\max}(H_{N,q}) = \Omega(\sqrt{N}/q)$ with q -dependence [6, Theorem I.3]; Theorem 13.10 identifies the sequential leading constant $\sqrt{2}$ and the first correction.

For orientation, the first-corrected approximation compares with numerical fixed- q values as follows:

q	$\sqrt{2}/q$	first-corrected	numerical E_q	relative error
4	0.35355	0.32217	0.32504	−0.88%
6	0.23570	0.22175	0.22267	−0.41%
8	0.17678	0.16893	0.16933	−0.24%
10	0.14142	0.13640	0.13661	−0.15%
12	0.11785	0.11436	0.11449	−0.11%

The $q = 4$ entry is He's value; the remaining entries are numerical Schwinger–Dyson consistency checks, displayed to the five significant figures that the extrapolation resolves. Figure 1 plots the same comparison.

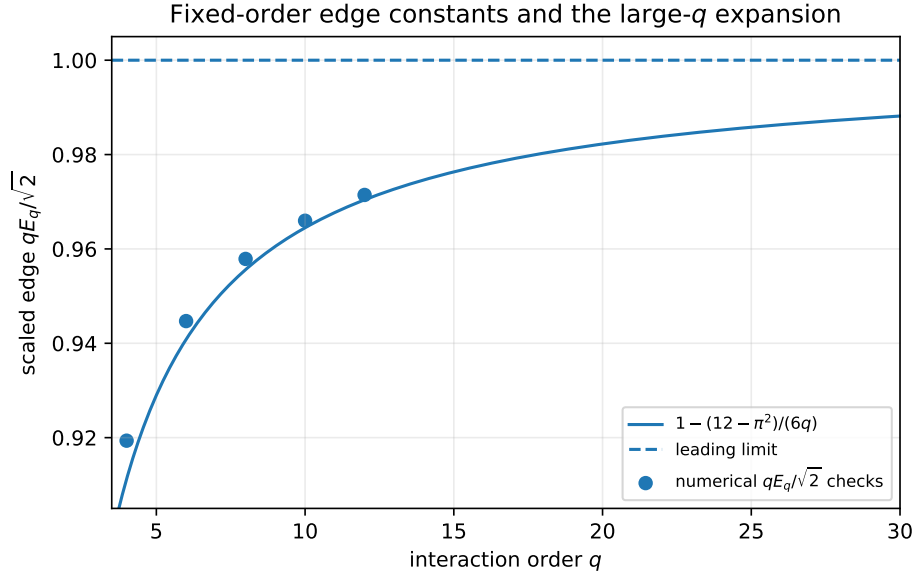


Figure 1: Numerical fixed- q consistency checks in the scaled variable $qE_q/\sqrt{2}$. The solid line is $1 - (12 - \pi^2)/(6q)$; the dashed line is the leading limit 1.

A Computational verification and proof dependencies

This work — research, proof development, verification, writing, and review — was produced with extensive use of AI systems, including large language models. Their tasks included source comparison, symbolic algebra, finite combinatorial enumeration, numerical consistency checks, cross-reference checking, and editorial review. All mathematical arguments are presented in this manuscript or attributed to cited results; the main theorems do not depend on numerical computation.

A.1 Supplementary consistency checks

The following checks were used to detect normalization, sign, combinatorial, and implementation errors during the preparation of the paper.

Check	Outcome
Majorana algebra and normalization	Hermiticity, unitarity, commutation signs, the standardized/extensive conversion, the Lehmann mass 2^{2-q} , and the cavity factor 2^q were checked symbolically for $q = 4, 6, 8, 10, 12$.
Sextic repeated-flip combinatorics	All set partitions of six ordered positions were enumerated, reproducing the coefficients in Lemma 6.2.
Fermionic cavity signs	Every pairing with at most four chords (124 pairings in total) was evaluated by exact Pauli-string arithmetic and agreed with $(-1)^{\text{cr}(\mathfrak{p})}$.
Large- q identities	The rescaling, transform constants, global logarithm bounds, Gamma-certificate coefficients, and the identity $\int_0^\infty D_\alpha(y)^2 dy = \pi/[2(1+2\alpha)]$ were checked independently by symbolic algebra and high-precision quadrature.
Independent Schwinger–Dyson implementations	The quantity compared is the finite-temperature grid estimator $E_q^{(\beta,K)} := \frac{2^{q/2+1}}{\sqrt{q}} \frac{\beta}{2K} \sum_{j=1}^K g_\beta(t_j)^q$, the midpoint discretization on the K -point grid t_j of the finite- β edge functional $\frac{2^{q/2+1}}{\sqrt{q}} \int_0^{\beta/2} g_\beta(t)^q dt$, whose $\beta, K \rightarrow \infty$ extrapolation is the estimate E_6^{num} below. At $q = 6$, $\beta = 16$, and $K = 128$, direct 80-digit summation gave $E_6^{(16,128)} = 0.2120722729381016735\dots$, while an independent DST-IV implementation differed by 2.5×10^{-15} .
Finite-dimensional Clifford checks	Random finite matrices were checked for Hermiticity and, for $q \equiv 2 \pmod{4}$, the exact spectral pairing predicted by Lemma 11.3.

The extrapolated zero-temperature standardized edge estimate

$$E_6^{\text{num}} \approx 0.2226690004$$

and the numerical values displayed in Section 13 are consistency checks; no rigorous error interval is asserted for these decimals.

A.2 Proof-critical external inputs

Most of the argument is proved directly in the preceding sections. The principal specialized external input is the operator-algebraic transfer mechanism developed for the quartic model by He [4]. The table

below records the imported results that are most consequential for the proof and the location where their model-dependent hypotheses are verified here.

External input	Role	Verification in this paper
He, Lemmas 2.2–2.4 [4]	Positivity, continuity, second-moment identity, and anti-monotonicity of the Dyson map	Lemma 4.2 states the arity-independent form used throughout.
He, results 4.12–4.19 [4]	Sign-system identities, relative-modular compactness, common support, and modularly invariant coordinate algebras	The sign identities are algebraic; compact-time commutator decay and the uniform field modulus are established in Theorem 6.7 and Lemma 6.8. These fixed-mark conclusions are isolated in Proposition 12.3(a).
He, results 5.1–5.12 [4]	Projective de Finetti law and complex-contour factorization for the two-occurrence incidence class, along a slow marked-site diagonal	The sextic deletion estimates, slow marked-site diagonal, and the incidence restrictions are verified in Theorem 6.10. These growing-mark conclusions are isolated in Proposition 12.3(b).
Raynaud [9, Proposition 2.2(i)]	Identification of compressed finite-dimensional cocycles in a Groh–Raynaud ultraproduct	Equal adjacent supports are obtained before the result is invoked in Proposition 6.9.
Takesaki [10, 11]	Conditional expectations and measurable direct-integral reduction	Applied only on the separably acting finite-coordinate algebra constructed in Proposition 6.9.
Standard compactness, Fourier, and concentration theorems	Existence of subsequential limits, transform inversion, Gaussian concentration, and Borel–Cantelli	Their hypotheses are verified at the point of use; see, in particular, Propositions 4.3 and 11.1 and Theorems 9.3–9.4.

The new Majorana phases, positive-Lehmann masses, deletion and locality counts, cavity signs, interpolation coefficients, zero-temperature estimates, and concentration bounds are established explicitly before the arity-neutral transfer results are invoked. The large- q argument in Section 13 is independent of the modular-transfer machinery once the fixed- q variational formula has been obtained.

A.3 Scope of the argument

- 1. Contour factorization.** Product disintegration is used only for the two-occurrence incidence class needed by the cavity expansion: distinct nondistinguished coordinates, two sign changes per coordinate, and no contraction between two such coordinates.
- 2. Zero-temperature passage.** Uniform tails control both the Dyson transform and the interaction energy, while the large-frequency expansion recovers the exact second moment after weak convergence rather than relying only on lower semicontinuity.
- 3. Almost-sure convergence.** The Gaussian Lipschitz estimate gives a summable e^{-cN^2} tail at deviations of order $\sqrt{N}$, so the almost-sure statement follows from the first Borel–Cantelli lemma and does not require independence across N .

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